Soliton cellular automaton, Toda molecule equation and sorting algorithm

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Abstract

A direct connection between a soliton cellular automaton(SCA) and an ultra-discrete analogue of the Toda molecule equation(uTM equation) is clarified. A solution to the SCA is presented by means of that to the uTM equation. A sorting algorithm based on this connection is also constructed.

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In 1990, one of the authors(D.T.) and Satsuma proposed a new-type of cellular automaton(or CA) called the "soliton cellular automaton"(or SCA) [1, 2]. It is made of infinite number of "0"'s(or boxes) and finite number of "1"'s(or balls) and is also called "box and ball system". A remarkable feature of the SCA is that any state consists only of solitons, interacting in the same manner as KdV solitons(Figure 1). It possesses infinitely many conserved quantities [3]. Recently, it has been clarified that integrable CA's are obtained from a limiting procedure of discrete integrable equations [4, 5]. This limiting procedure, which is called the "ultra-discrete limit", is applied to various kind of soliton equations [6, 7, 8]

$t = 0$: ··· 0 1 1 1 1 0 0 0 0 0 1 1 0 0 1 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 ···

$t = 1$: ··· 0 0 0 0 0 1 1 1 1 0 0 0 1 1 0 1 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 ···

$t = 2$: ··· 0 0 0 0 0 0 0 0 0 1 1 1 0 0 1 0 1 1 1 0 0 0 0 0 0 0 0 0 0 0 0 0 0 ···

$t = 3$: ··· 0 0 0 0 0 0 0 0 0 0 0 0 1 1 0 1 0 0 0 1 1 1 1 0 0 0 0 0 0 0 0 0 0 ···

$t = 4$: ··· 0 0 0 0 0 0 0 0 0 0 0 0 0 0 1 0 1 1 0 0 0 0 0 1 1 1 1 0 0 0 0 0 0 ···

$t = 5$: ··· 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 1 0 0 1 1 0 0 0 0 0 0 0 1 1 1 1 0 0 ···

Figure 1: Soliton cellular automaton(3-soliton)

In the present paper [9], we have applied the ultra-discrete limit to the Toda molecule equation, which we call the uTM equation in this context. The Toda molecule equation posseses the so-called "molucular-type solutions", which converge to a certain limit in their time evolution, and is considered to be different from other soliton equations.

The aim of this paper is to show an equivalence between the above SCA and uTM equation. We also give an exact solution to the SCA, and mention a connection between conserved quantities of the uTM equation and those of SCA constructed through a combinatoric approach [3]. We finally present a sorting algorithm by employing this equivalence.

We first introduce dependent variables (Q_n^t, E_n^t) in relation to the SCA. Let Q_n^t be number of "1"'s in the n-th block of "1"(or 1-block, for short) and E_n^t be number of "0"'s in the n-th block of "0" (or 0-block) with boundary conditions $E_0^t = +\infty,\ E_N^t = +\infty$, where N is number of solitons. For example, in Figure 1, we have

$$Q_1^0 = 4,\ Q_2^0 = 2,\ Q_3^0 = 1,\ E_0^0 = +\infty,\ E_1^0 = 5,\ E_2^0 = 2,\ E_3^0 = +\infty.$$

Then the quantities (Q_n^t, E_n^t) satisfy following equations;

$$Q_n^{t+1} = \min(E_n^t, \sum_{k=1}^{n} Q_k^t - \sum_{k=1}^{n-1} Q_k^{t+1}) \quad (1 \le n \le N), \tag{1}$$

$$E_n^{t+1} = E_n^t + Q_{n+1}^t - Q_n^{t+1} \quad (1 \le n \le N-1), \tag{2}$$

$$E_0^t = E_N^t = +\infty. \tag{3}$$

Since Q_n^{t+1}(= the size of the n-th 1-block at time $t+1$) is determined by the minimum between E_n^t(= the size of the n-th 0-block at time t) and $\sum_{k=1}^{n} Q_k^t - \sum_{k=1}^{n-1} Q_k^{t+1}$(= the number of "1"'s which have the possibility to enter the n-th 0-block from time t to $t+1$), eq. (1) is clear. Equation (2) also holds due to the fact that the first "$\underline{0}$" in the n-th 0-block turns to the first "$\underline{1}$" in the n-th 1-block and the last "$\underline{1}$" in the $(n+1)$-th 1-block to the last "$\underline{0}$" in the n-th 0-block from time t to $t+1$, whether or not the n-th 1-block interacts with others(Figure 2, 3).

$$t\colon \cdots 0\ 1\ 1\ 1\ 1\ 1\ \underline{0}\ 0\ 0\ 1\ 1\ \underline{1}\ \underline{0}\ 0\ 0\ \underline{1}\ 0\ 0\ 0\ 0 \cdots$$

$$t+1\colon \cdots 0\ 0\ 0\ 0\ 0\ 0\ \underline{1}\ 1\ 1\ 0\ 0\ \underline{0}\ \underline{1}\ 1\ 1\ \underline{0}\ 1\ 1\ 1\ 0 \cdots$$

Figure 2: Interacting case

$$t\colon \cdots 0\ 1\ 1\ 1\ 1\ \underline{0}\ 0\ 0\ 0\ 0\ 0\ 1\ \underline{1}\ 0\ 0\ 0\ 0\ 0\ 0\ 0 \cdots$$

$$t+1\colon \cdots 0\ 0\ 0\ 0\ 0\ \underline{1}\ 1\ 1\ 1\ 0\ 0\ 0\ 0\ \underline{0}\ 1\ 1\ 0\ 0\ 0\ 0 \cdots$$

Figure 3: Non-interacting case

Equations (1)-(3) are considered as an ultra-discrete analogue of the Toda molecule equation (uTM equation) [9, 10]. These equations are obtained by taking the limit $\varepsilon \to +0$ for the equations,

$$\begin{aligned}
\varepsilon \log(\mathrm{e}^{-Q_n^{t+1}/\varepsilon}) &= \varepsilon \log\left(\mathrm{e}^{-E_n^t/\varepsilon} + \frac{\mathrm{e}^{-Q_n^t/\varepsilon}\mathrm{e}^{-Q_{n-1}^t/\varepsilon}\cdots\mathrm{e}^{-Q_1^t/\varepsilon}}{\mathrm{e}^{-Q_{n-1}^{t+1}/\varepsilon}\mathrm{e}^{-Q_{n-2}^{t+1}/\varepsilon}\cdots\mathrm{e}^{-Q_1^{t+1}/\varepsilon}}\right) &(4)\\
\varepsilon \log(\mathrm{e}^{-E_n^{t+1}/\varepsilon}) &= \varepsilon \log\left(\frac{\mathrm{e}^{-Q_{n+1}^t/\varepsilon}\mathrm{e}^{-E_n^t/\varepsilon}}{\mathrm{e}^{-Q_n^{t+1}/\varepsilon}}\right). &(5)
\end{aligned}$$

The above equations are equivalent to

$$\begin{aligned}
I_n^{t+1} &= V_n^t + \frac{I_n^t I_{n-1}^t \cdots I_1^t}{I_{n-1}^{t+1} I_{n-2}^{t+1} \cdots I_1^{t+1}}, &(6)\\
V_n^{t+1} &= \frac{I_{n+1}^t V_n^t}{I_n^{t+1}} &(7)
\end{aligned}$$

by introducing dependent variables $I_n^t = \mathrm{e}^{-Q_n^t/\varepsilon}$, $V_n^t = \mathrm{e}^{-E_n^t/\varepsilon}$. By employing eq. (7) with boundary conditions $V_0^t = V_N^t = 0$, eq. (6) is rewritten as

$$I_n^{t+1} = I_n^t + V_n^t - V_{n-1}^{t+1}, \tag{8}$$

which is, together with eq. (7), nothing but the time-discrete Toda molecule equation [11].

A solution to the SCA can also be constructed by means of that to the uTM equation [9]. This is given by

$$
\begin{aligned}
Q_n^t \; = \; & \min_{i_1<\cdots<i_{n-1}} \left(\sum_{k=1}^{n-1} 2(n-1-k)P_{i_k} + C_{i_k} + tP_{i_k} \right) \\
& + \min_{i_1<\cdots<i_n} \left(\sum_{k=1}^{n} 2(n-k)P_{i_k} + C_{i_k} + (t+1)P_{i_k} \right) \\
& - \min_{i_1<\cdots<i_n} \left(\sum_{k=1}^{n} 2(n-k)P_{i_k} + C_{i_k} + tP_{i_k} \right) \\
& - \min_{i_1<\cdots<i_{n-1}} \left(\sum_{k=1}^{n-1} 2(n-1-k)P_{i_k} + C_{i_k} + (t+1)P_{i_k} \right), \qquad (9) \\
E_n^t \; = \; & \min_{i_1<\cdots<i_{n+1}} \left(\sum_{k=1}^{n+1} 2(n+1-k)P_{i_k} + C_{i_k} + tP_{i_k} \right) \\
& + \min_{i_1<\cdots<i_{n-1}} \left(\sum_{k=1}^{n-1} 2(n-1-k)P_{i_k} + C_{i_k} + (t+1)P_{i_k} \right) \\
& - \min_{i_1<\cdots<i_n} \left(\sum_{k=1}^{n} 2(n-k)P_{i_k} + C_{i_k} + tP_{i_k} \right) \\
& - \min_{i_1<\cdots<i_n} \left(\sum_{k=1}^{n} 2(n-k)P_{i_k} + C_{i_k} + (t+1)P_{i_k} \right), \qquad (10)
\end{aligned}
$$

where P_i's stand for the size of N solitons and C_i's are parameters determined by the initial condition. The parameters P_i's satisfy, without loss of generality, an inequality,

$$P_1 < P_2 < \cdots < P_N. \qquad (11)$$

We have, in the limit $t \to \pm\infty$,

$$Q_n^t \;\to\; P_{N-n+1} \; (t \to -\infty), \qquad (12)$$

$$Q_n^t \;\to\; P_n \; (t \to +\infty), \qquad (13)$$

which reflects the fact that N solitons are arranged in the decreasing and increasing order in their size as $t \to -\infty$ and $t \to +\infty$, respectively.

We next mention the relation between conserved quantities of the uTM eq. [9] and those of the SCA [3]. Conserved quantities for eqs. (1)-(3) are given by

$$
\begin{aligned}
uC_1 &= \min\left(\min_{1\le k_1\le N} Q^t_{k_1}, \min_{1\le l_1\le N-1} E^t_{l_1}\right), &(14)\\
uC_2 &= \min\Bigg[\min_{1\le k_1<k_2\le N}(Q^t_{k_1}+Q^t_{k_2}), \min_{\substack{1\le k_1\le N,1\le l_1\le N-1\\ k_1\notin\{l_1,l_1+1\}}}(Q^t_{k_1}+E^t_{l_1}),\\
&\qquad \min_{1\le l_1<l_2\le N-1}(E^t_{l_1}+E^t_{l_2})\Bigg], &(15)\\
uC_3 &= \min\Bigg[\min_{1\le k_1<k_2<k_3\le N}(Q^t_{k_1}+Q^t_{k_2}+Q^t_{k_3}), \min_{\substack{1\le k_1<k_2\le N,1\le l_1\le N-1\\ \{k_1,k_2\}\cap\{l_1,l_1+1\}=\phi}}(Q^t_{k_1}+Q^t_{k_2}+E^t_{l_1}),\\
&\qquad \min_{\substack{1\le k_1\le N,1\le l_1<l_2\le N-1\\ k_1\notin\{l_1,l_1+1,l_2,l_2+1\}}}(Q^t_{k_1}+E^t_{l_1}+E^t_{l_2}), \min_{1\le l_1<l_2<l_3\le N-1}(E^t_{l_1}+E^t_{l_2}+E^t_{l_3})\Bigg], &(16)\\
&\vdots\\
uC_m &= \min_{\substack{0\le i,j\le m\\ i+j=m}}\left[\min_{\substack{1\le k_1<\cdots<k_i\le N,1\le l_1<\cdots<l_j\le N-1\\ \{k_1,\cdots,k_i\}\cap\{l_1,l_1+1,\cdots,l_j,l_j+1\}=\phi}}(Q^t_{k_1}+\cdots+Q^t_{k_i}+E^t_{l_1}+\cdots+E^t_{l_j})\right], &(17)\\
&\vdots\\
uC_N &= Q^t_1+Q^t_2+\cdots+Q^t_N. &(18)
\end{aligned}
$$

Taking the limit $t \to +\infty$ and considering the fact that each uC_m is time-independent, we see that the relations,

$$uC_m = P_1 + P_2 + \cdots + P_m \quad (m = 1, 2, \cdots, N) \tag{19}$$

hold. On the other hand, Torii et.al [3] proved that the shape of the Young diagram, which is constructed from the SCA through combinatoric methods, is time-independent and that the length of the i-th column from the right is equal to that of the i-th smallest soliton P_i. This fact, together with eq. (19), indicates that each uC_m corresponds to the size of subset of the Young diagram which consists of m columns from the right(Figure 4).

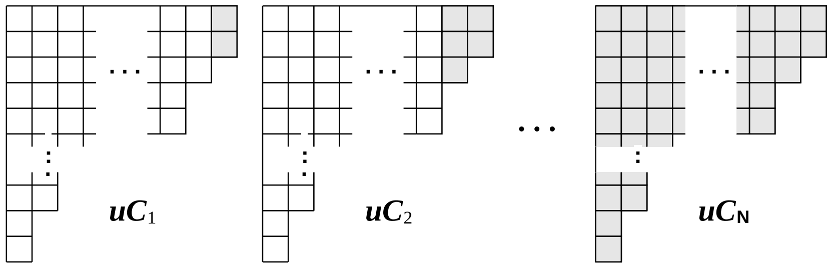

Figure 4: Conserved quantities for eqs. (1)-(3) and the Young diagram

Finally, we accelerate sorting based on the uTM equation. We have recently shown [9] that the uTM equation possesses a sorting property and that a sorting algorithm can be constructed by using the equation. Its numerical example is given in table 1. However, this algorithm has a serious defect. As one can see from the table 1, it requires many time steps until the numerals "4" and "3" are replaced with each other(Q_4^t and Q_5^t at $t = 21, \cdots, 32$). This fact is interpreted, in terms of the SCA, as follows. Since the value of E_4^t is equal to 14 at $t = 21$, its corresponding SCA is partially drawn as

$$t = 21 : \cdots 0 \ \underbrace{1\,1\,1\,1}_{4} \ \underbrace{0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0}_{14} \ \underbrace{1\,1\,1}_{3} \ 0 \cdots$$

As can be seen from above, it takes many time steps until the block "1111" catches up with "111". In order to cut down time steps, we get rid of extra "0"'s(or empty boxes) as follows.

$$t=21 \quad : \quad \cdots 0\ \underbrace{1\ 1\ 1\ 1}_{4}\ \overbrace{\not0\not0\not0\not0\not0\not0\not0\not0\not0\not0\not0}^{11}\ \underbrace{0\ 0\ 0}_{3}\ \underbrace{1\ 1\ 1}_{3}\ 0\cdots$$

$$\Downarrow$$

$$t=21 \quad : \quad \cdots 0\ \underbrace{1\ 1\ 1\ 1}_{4}\ \underbrace{0\ 0\ 0}_{3}\ \underbrace{1\ 1\ 1}_{3}\ 0\cdots$$

At the next time step $t=22$, the blocks "1111" and "111" are replaced with each other. In other words, the numerals "4" and "3" are sorted in the increasing order. Taking the same procedures, that is, resetting the value of E_n^t as Q_{n+1}^t, at each time step, we can derive a modified sorting algorithm. A numerical example of the modified algorithm is given in table 2, from which we see that time steps required for sorting is reduced.

The above modification is written, in terms of equation, as follows. Substitution $E_n^t = Q_{n+1}^t$ into one of the uTM eq. (1) yields

$$Q_n^{t+1} = \min(Q_{n+1}^t, \sum_{k=1}^{n} Q_k^t - \sum_{k=1}^{n-1} Q_k^{t+1}) \quad (1 \le n \le N). \tag{20}$$

The above equation is nothing but the bubble sort, one of the most fundamental sorting algorithms with $O(N^2)$ time steps required for sorting.

We have shown the equivalence between the soliton cellular automaton and ultra-discrete limit of the Toda molecule equation. It is interesting that soliton and molecular-type solutions correspond with each other in their ultra-discrete limits. This equivalence also reminds us of the recent paper by A. Nakamura [12], in which a one-soliton solution to the Toda molecule equation is obtained. By changing the auxiliary variable E_n^t, we can accelerate sorting based on the uTM equation. The bubble sort is derived by modifying the uTM equation. However, it is still unknown whether the bubble sort, when viewed as an ultra-discrete equation, is integrable, in other words, possesses any exact solution. It is also interesting to establish new sorting algorithms, which require less time steps, from the viewpoint of ultra-discrete integrable systems.

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Table 1: The sorting algorithm based on the uTM equation

t	Q_1^t	Q_2^t	Q_3^t	Q_4^t	Q_5^t	Q_6^t	Q_7^t	Q_8^t	Q_9^t	Q_{10}^t	Q_{11}^t
0	24	−20	4	18	3	−16	10	6	7	29	−14
1	−20	4	18	3	−16	10	6	7	24	−14	29
2	−20	4	18	3	−16	10	6	7	24	−14	29
3	−20	4	3	−16	10	6	7	18	−4	14	29
4	−20	4	3	−16	10	6	7	18	−14	24	29
5	−20	4	−16	3	6	7	10	8	−4	24	29
6	−20	4	−16	3	6	7	10	−14	18	24	29
7	−20	4	−16	3	6	7	10	−14	18	24	29
8	−20	−10	−2	3	6	7	0	−4	18	24	29
9	−20	−16	4	3	6	7	−14	10	18	24	29
10	−20	−16	4	3	6	7	−14	10	18	24	29
11	−20	−16	4	3	6	−12	5	10	18	24	29
12	−20	−16	4	3	6	−14	7	10	18	24	29
13	−20	−16	4	3	−6	−2	7	10	18	24	29
14	−20	−16	4	3	−14	6	7	10	18	24	29
15	−20	−16	4	3	−14	6	7	10	18	24	29
16	−20	−16	4	3	−14	6	7	10	18	24	29
17	−20	−16	4	−2	−9	6	7	10	18	24	29
18	−20	−16	4	−14	3	6	7	10	18	24	29
19	−20	−16	4	−14	3	6	7	10	18	24	29
20	−20	−16	−6	−4	3	6	7	10	18	24	29
21	−20	−16	−14	4	3	6	7	10	18	24	29
22	−20	−16	−14	4	3	6	7	10	18	24	29
23	−20	−16	−14	4	3	6	7	10	18	24	29
24	−20	−16	−14	4	3	6	7	10	18	24	29
25	−20	−16	−14	4	3	6	7	10	18	24	29
26	−20	−16	−14	4	3	6	7	10	18	24	29
27	−20	−16	−14	4	3	6	7	10	18	24	29
28	−20	−16	−14	4	3	6	7	10	18	24	29
29	−20	−16	−14	4	3	6	7	10	18	24	29
30	−20	−16	−14	4	3	6	7	10	18	24	29
31	−20	−16	−14	4	3	6	7	10	18	24	29
32	−20	−16	−14	4	3	6	7	10	18	24	29
33	−20	−16	−14	3	4	6	7	10	18	24	29

Table 2: The sorting algorithm based on the uTM equation and the SCA

t	Q^t_1	Q^t_2	Q^t_3	Q^t_4	Q^t_5	Q^t_6	Q^t_7	Q^t_8	Q^t_9	Q^t_{10}	Q^t_{11}
0	24	−20	4	18	3	−16	10	6	7	29	−14
1	−20	4	18	3	−16	10	6	7	24	−14	29
2	−20	4	3	−16	10	6	7	18	−14	24	29
3	−20	3	−16	4	6	7	10	−14	18	24	29
4	−20	−16	3	4	6	7	−14	10	18	24	29
5	−20	−16	3	4	6	−14	7	10	18	24	29
6	−20	−16	3	4	−14	6	7	10	18	24	29
7	−20	−16	3	−14	4	6	7	10	18	24	29
8	−20	−16	−14	3	4	6	7	10	18	24	29