THE ULTIMATE DISCRETISATION OF THE PAINLEVÉ EQUATIONS

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Abstract

We present a systematic way to construct ultra-discrete versions of the Painlevé equations starting from known discrete forms. These ultra-discrete equations are cellular automata the sense that the dependent variable takes only integer values. The ultra-discrete Painlevé equations have properties characteristic of the continuous and discrete Painlevé's like coalescence cascades, particular solutions and auto-Bäcklund relations.

1. Introduction

What does make the Painlevé equations so important as integrable systems? More than any other property it is their omnipresence. Whenever one studies integrability, under any of its disguises, one is bound to encounter, sooner or later, Painlevé equations. They are the integrable archetypes, being the simplest, genuinely nonlinear, nontrivially integrable systems. Their integrable character manifests itself in several ways. They have a very particular structure of singularities: all their movable (initial-condition-dependent) singularities are poles. In fact, this is how they were discovered and this special singularity structure (Painlevé property) became a criterion for integrability. They possess rich families of particular solutions for specific values of their parameters. There exist a host of interrelations relating one Painlevé equation to some other or one to itself (for different values of the parameters). Many other properties do exist and it is not clear whether one can give an exhaustive list.

Painlevé equations have been discovered, in a fully continuous setting, as second order differential equations. Higher order Painlevé equations surely exist but only the second order ones have been identified and thoroughly studied. Recent developments in the domain of integrability have led to spectacular progress concerning Painlevé equations, in particular with the discovery of their discrete analogs. The latter were shown to possess all the properties of their continuous counterparts. The analog of the Painlevé property here is the notion of singularity confinement, which has become an integrability criterion for discrete systems. Moreover, it is getting progressively clearer that these discrete objects are more fundamental than the continuous ones. As a matter of fact, the latter can be obtained as limiting cases of the former. Semi-continuous Painlevé equations have also been obtained. They present the peculiarity that they can be interpreted in two different ways: either as differential-difference systems or as differential-delay ones. However their study is not as complete as that of the fully discrete equations.

In this paper we shall investigate Painlevé equations in a still new domain: that of the ultra-discrete systems. This name is used to designate systems where the dependent variables, as well as the independent ones, take only discrete values. In this respect ultra-discrete systems are cellular automata. However we reserve the name of ultra-discrete to systems obtained from discrete ones through a limiting procedure introduced in [] and which will be detailed in the next section. Thus the ultra-discrete Painlevé equations will be obtained in a systematic, algorithmic way, starting from the appropriate well-known discrete forms. In what follows we shall show that ultra-discrete forms exist for all six Painlevé equations. Moreover the equations obtained have properties that are characteristic of Painlevé equations. They are organized in coalescence cascades, possess rich particular solutions (globally described) and auto-Bäcklund transformations that relate the solutions of the same equation corresponding to different values of the parameters. We discuss also the possible integrability criteria for ultra-discrete systems.

2. The ultra-discrete limit

Before introducing the ultra-discrete limit let us first consider the question of nonlinearity. How simple can a nonlinear system be and still be *genuinely* non linear. The nonlinearities we are accustomed to, involving powers, are not necessarily the simplest. It turns out (admittedly with hindsight) that the simplest nonlinear function of x one can think of is $|x|$. It is indeed linear for *both* $x > 0$ and $x < 0$ and the nonlinearity comes only from the different determinations. Thus one would expect the equations involving nonlinearities only in terms of absolute-values to be the simplest.

The ultra-discrete limit does just that, i.e. it converts a given (discrete) nonlinear equation to one where only absolute-value nonlinearities appear. The key relation is the following limit:

$$\lim_{\epsilon \to 0} \epsilon \log(1 + e^{x/\epsilon}) = \max(0, x) = (x + |x|)/2. \tag{2.1}$$

Other equivalent expressions exist for this limit and the notation that is often used is the truncated power function $(x)_+ \equiv \max(0, x)$. It is easy to show that $\lim_{\epsilon \to 0} \epsilon \log(e^{x/\epsilon} +$

$e^{y/\epsilon}) = \max(x, y)$ and the extension to n terms in the argument of the logarithm is straightforward.

Let us give, as an example, the ultra-discretisation of the (potential) mKdV equation. We start with the discrete form:

$$x_n^{k+1} = x_n^{k-1} \frac{x_{n+1}^k + \mu x_{n-1}^k}{\mu x_{n+1}^k + x_{n-1}^k} \tag{2.2}$$

First we divide the numerator and denominator of the rhs by x_{n-1}^k and take the logarithm of both sides. Next we introduce X through $x = e^{X/\epsilon}$ (and also $\mu = e^{m/\epsilon}$) and take the limit $\epsilon \to 0$. Using (2.1) we find:

$$X_n^{k+1} = m + X_n^{k-1} + (X_{n+1}^k - X_{n-1}^k - m)_+ - (X_{n+1}^k - X_{n-1}^k + m)_+ \; eqno(2.3)$$

which is the ultra-discrete form of mKdV.

Two remarks are in order at this point. First, since the function $(x)_+$ takes only integer values when the argument is integer, the ultra-discrete equations can describe cellular automata (CA), provided one restricts the initial conditions to integer values. This approach has already been used in order to introduce CA related to many interesting evolution equations. Second, the necessary condition for the procedure to be applicable is that the dependent variables be positive, since we are taking a logarithm. This means that only some solutions of the discrete equations will survive in the ultra-discretisation. Moreover, the limit (2.1) exists only if the two terms in the argument of the logarithm have positive sign. While this is only a technical difficulty, it can sometimes limit the application of the method.

3. Ultra-discrete Riccati equations

Before embarking upon studying the ultra-discrete form of the Painlevé equations it is intersting to examine the case of the Riccati equation. Although the latter does not define a genuine transcendent, it is still the simplest integrable nonlinear equation. Its discrete form is known. It is just the homographic mapping and writes:

$$x_{n+1} = \frac{\alpha x_n + \beta}{\gamma x_n + \delta} \tag{3.1}$$

In order to proceed to its ultradiscretisation, let us consider the constraints posed by the positivity requirement we discussed in the previous section. Taking all of $\alpha, \beta, \gamma, \delta$ positive and $0 < x_n < \infty$, we find that the domain of variation of x_{n+1} is from β/d to α/γ. Thus, if we are considering the inverse evolution from x_{n+1} to x_n, and we are given $0 < x_{n+1} < \infty$ it is clear that we will not find in general a positive preimage x_n to x_{n+1}. The only way to ensure this is to have x_{n+1} take values in the full range $(0, \infty)$ and this is possible only if either $\beta = \gamma = 0$ (i.e. a linear mapping) or $\alpha = \delta = 0$. In the latter case the Riccati becomes simply $x_{n+1} = \beta/x_n$ (with $\gamma = 1$).

Thus the only homographic mappings that are ultra-discretisable are the mappings:

$$x_{n+1} = \alpha x_n \tag{3.2}$$

$$x_n x_{n+1} = \beta \tag{3.3}$$

The ultra-discrete limit is straightforward. We introduce $x = e^{X/\epsilon}$, $\alpha = e^{a/\epsilon}$, $\beta = e^{b/\epsilon}$, and take the logarithm of both sides of the equation. We find readily the ultra-discrete forms:

$$X_{n+1} = X_n + a \tag{3.4}$$

$$X_{n+1} + X_n = b \tag{3.5}$$

Thus the Riccati equation (3.5) becomes *linear* at the ultra-discrete limit. This is quite satisfactory, since it retains the essential feature of the Riccati, namely to be a linearizable equation.

4. The ultra-discrete Painlevé equations and their coalescence cascade

In order to proceed to the analysis of the Painlevé equations, let us start by listing their appropriate (multiplicative) discrete forms:

d-$\mathrm{P}_{\mathrm{I}-1}$

$$x_{n+1}x_{n-1} = \frac{\alpha\lambda^n}{x_n} + \frac{1}{x_n^2} \tag{4.1}$$

d-$\mathrm{P}_{\mathrm{I}-2}$

$$x_{n+1}x_{n-1} = \alpha\lambda^n + \frac{1}{x_n} \tag{4.2}$$

d-$\mathrm{P}_{\mathrm{I}-3}$

$$x_{n+1}x_{n-1} = \alpha\lambda^n x_n + 1 \tag{4.3}$$

d-$\mathrm{P}_{\mathrm{II}-1}$

$$x_{n+1}x_{n-1} = \frac{\alpha\lambda^n(x_n + \lambda^n)}{x_n(1 + x_n)} \tag{4.4}$$

d-$\mathrm{P}_{\mathrm{II}-2}$

$$x_{n+1}x_{n-1} = \frac{x_n + \alpha\lambda^n}{1 + \beta x_n\lambda^n} \tag{4.5}$$

d-$\mathrm{P}_{\mathrm{III}}$

$$x_{n+1}x_{n-1} = \frac{(x_n + \alpha\lambda^n)(x_n + \beta\lambda^n)}{(1 + \gamma x_n\lambda^n)(1 + \delta x_n\lambda^n)} \tag{4.6}$$

The remaining Painlevé IV, V and VI equations are best written as systems for two components:

d-$\mathrm{P}_{\mathrm{IV}-1}$

$$x_{n+1}x_n = 1 + y_n \tag{4.7a}$$

$$y_n y_{n-1} = \frac{x_n^2 + (\alpha + 1/\alpha)x_n + 1}{\lambda^n(1 + \beta\lambda^n x_n)} \tag{4.7b}$$

d-$\mathrm{P}_{\mathrm{IV}-2}$

$$x_{n+1}x_n = 1 + y_n \tag{4.8a}$$

$$y_n y_{n-1} = \frac{(x_n^2 + (\alpha + 1/\alpha)x_n + 1)(1 + \beta\lambda^n x_n)}{1 + \gamma\lambda^{2n} x_n} \tag{4.8b}$$

d-P_{V}

$$x_{n+1}x_n = 1 + y_n \tag{4.9a}$$

$$y_n y_{n-1} = \frac{(x_n^2 + (\alpha + 1/\alpha)x_n + 1)(x_n^2 + (\beta + 1/\beta)x_n + 1)}{(1 + \gamma\lambda^n x_n)(1 + \delta\lambda^n x_n)} \tag{4.9b}$$

d-P_{VI}

$$x_{n+1}x_n = \frac{(y_n + \alpha\lambda^n)(y_n + \beta\lambda^n)}{(1 + \gamma y_n \lambda^n)(1 + \delta y_n \lambda^n)} \tag{4.10a}$$

$$y_n y_{n-1} = \frac{(x_n + \zeta\lambda^n)(x_n + \eta\lambda^n)}{(1 + \theta x_n \lambda^n)(1 + \kappa x_n \lambda^n)} \tag{4.10b}$$

with $\gamma\delta = \theta\kappa$ and $\alpha\beta = \lambda\zeta\eta$. Equation (4.9) is just the 'standard' form of d-P_V written as a system, equations (4.7-8) are its limiting and degenerate forms [] respectively while d-P_{VI} (4.10) is the asymmetric extension of d-P_{III} propose by Jimbo ans Sakai [].

Let us point out here that the equations are symmetric as far as the forward and the backward evolutions are concerned. Thus the positivity requirement once enforced in one direction is automatically guaranteed in the reverse one. We shall not go into the details of the ultra-discretisation procedure. The method outlined in Section 2 is applied in a straightforward way. We first introduce the variable transformation $x = e^{X/\epsilon}$, $y = e^{Y/\epsilon}$, and a similar transformation for all the constants that appear in the equation. In all multiplicative d-P's, the independent variable enters through $z = \lambda^n$. It suffices thus to put $\lambda = e^{1/\epsilon}$ and obtain $z = e^{n/\epsilon}$. This leads to the following list of the ultra-discrete Painlevé equations:

u-P_{I-1}

$$X_{n+1} + X_{n-1} + 2X_n = (X_n + n)_+ \tag{4.11}$$

u-P_{I-2}

$$X_{n+1} + X_{n-1} + X_n = (X_n + n)_+ \tag{4.12}$$

u-P_{I-3}

$$X_{n+1} + X_{n-1} = (X_n + n)_+ \tag{4.13}$$

u-P_{II-1}

$$X_{n+1} + X_{n-1} = a + (n - X_n)_+ - (n + X_n)_+ \tag{4.14}$$

u-$\mathrm{P}_{\mathrm{II}-2}$

$$X_{n+1} + X_{n-1} - X_n = a + (n - X_n)_+ - (n + X_n)_+ \tag{4.15}$$

u-$\mathrm{P}_{\mathrm{III}}$

$$X_{n+1}+X_{n-1}-2X_n = (n+a-X_n)_+ + (n-a-X_n)_+ - (X_n+b+n)_+ - (X_n-b+n)_+ \tag{4.16}$$

u-$\mathrm{P}_{\mathrm{IV}-1}$

$$X_{n+1} + X_n = (Y_n)_+ \tag{4.17a}$$

$$Y_n + Y_{n-1} = (X_n - a)_+ + (X_n + a)_+ - (X_n + b + n)_+ - n \tag{4.17b}$$

u-$\mathrm{P}_{\mathrm{IV}-2}$

$$X_{n+1} + X_n = (Y_n)_+ \tag{4.18a}$$

$$Y_n + Y_{n-1} = (X_n - a)_+ + (X_n + a)_+ - (X_n + b + 2n)_+ + (X_n - n)_+ \tag{4.18b}$$

u-P_{V}

$$X_{n+1} + X_n = (Y_n)_+ \tag{4.19a}$$

$$Y_n + Y_{n-1} = (X_n - a)_+ + (X_n + a)_+ + (X_n - b)_+ + (X_n + b)_+ - (X_n + c + n)_+ - (X_n - c + n)_+ \tag{4.19b}$$

u-P_{VI}

$$X_{n+1}+X_n-2Y_n = (2n+a-Y_n)_+ + (2n-a-Y_n)_+ - (2n+b+Y_n)_+ - (2n-b+Y_n)_+ \tag{4.20a}$$

$$Y_n+Y_{n-1}-2X_n = (2n-1+c-X_n)_+ + (2n-1-c-X_n)_+ - (2n-1+d+X_n)_+ - (2n-1-d+X_n)_+ \tag{4.20b}$$

We must point out here that the forms given above are the canonical ones. This means that we have performed all allowed transformations (translations of X and Y, and linear transformations of n) in order to eliminate redundant parameters and bring them into standard form. Moreovergiven the form of the equation, we can assume without loss of generality that the a, b, c in u-P_{III} are positive. The same applies to parameter a (but not b) in both u-P_{IV}'s, a, b, c in u-P_V and a, b, c, d in u-P_{VI}.

In order to derive the coalescence cascade of the u-P's we introduce a large parameter Ω. The coalescence limits are obtained through $\Omega \to +\infty$. In the cases of u-P_V and u-P_{IV} we assume that $Y_n > 0$ and, thus, $Y_n \equiv X_{n+1} + X_n$. As in the case of the continuous and discrete Painlevé equations P_V has two different coalescence limits. Putting $b = 2\Omega$, $c = \Omega - \alpha$, $n = \Omega + m - \alpha$ we find u-P_{IV-1} for the variable X_m at the limit $\Omega \to +\infty$. On the other hand, introducing $a = \Omega + \alpha$, $b = \Omega - \alpha$, $n = 2m - \Omega$, $X_n = \Omega + Z_m - m$, we find u-P_{III} for Z_m. Starting from u-P_{IV-1}, we put $a = \Omega + \alpha$, $n = 2m - \Omega - b - \alpha$, $X_n = \Omega + Z_m - m + \alpha$ and obtain u-P_{II-1} for Z_m. Similarly starting from u-P_{III} we can obtain u-P_{II-1} through $a = \Omega + \alpha$, $b = \Omega - \alpha$, $n = m + \Omega$, $X_n = Z_m - \alpha$. Finally from u-P_{II-1} we obtain u-P_{I-1} by putting $a = 4\Omega + \alpha$, $n = \Omega - m$ and $X_n = Z_m + \Omega$. It turns out that starting from u-P_{II-1} we can also obtain u-P_{I-2}. In this case we must take $a = \alpha - 2\Omega$, $2n = -3m - 2\Omega$ and $X_n = Z_m - m/2 - \Omega$.

No coalescence relation appears to exist between u-P_V and u-P_{IV-2}: the two equations do not belong to the same coalescence cascade already in the discrete case. Still u-P_{IV-2} is related to a u-P_{II}. In fact, putting $a = \Omega + \alpha$, $b = \Omega - \alpha$, $n = m - \Omega$, $X_n = \Omega + Z_m - m + \alpha$ in u-P_{IV-2} gives us u-P_{II-2}. From u-P_{II-2} we can get u-P_{I-2}, provided we put $a = 3\Omega + \alpha$, $n = \Omega - m$ and $X_n = \Omega + Z_m$. Just as in the case of u-P_{II-1}, here also we can find a second coalescence. Putting $a = -\Omega + \alpha$, $n = -\Omega - 2m$ and $X_n = Z_m - \Omega - m$ we obtain at the limit $\Omega \to +\infty$ equation u-P_{I-3}.

Finally, we must point out that u-P_{VI} is not related to the remaining u-P's through coalescence (as expected from what is known in the discrete case). The only relation one can find is between u-P_{VI} and u-P_{III} where the latter appears as a reduction of the former. Indeed taking $c = a$, $b = d$ and $Z_{2n-1} = X_n$, $Z_{2n} = Y_n$ in (4.20) allows one to recover (4.16) for Z_n.

5. Special solutions of the ultra-discrete Painlevé equations

The special solutions of the Painlevé equations play the same role as the multisoliton solutions for the integrable evolution equations. Solutions of both continuous and discrete Painlevé equations have been obtained (for particular values of their parameters) either rational or in terms of special functions. Special solutions exist also for the ultra-discrete Painlevé equations (except for u-$\mathrm{P_I}$ which is parameter-free).

Let us start with u-$\mathrm{P}_{\mathrm{II}-1}$. We find readily that for $a = 0$, a solution $X = 0$ exists. The next solution for (4.14) is a step-function one. Indeed when n is large, a constant positive solution exists for $a = 4p$, with integer p, where X is equal to p while a constant solution with $X = 2p$ exists when n is large negative. The remarkable fact is that that these constant "half" solutions do really join to form a solution of (4.14) with p jumps from the value $X = 2p$ to $X = p$. The first jump occurs at $n_0 = 1 - 2|p|$ and we have successive jumps of $-|p|/p$ at $n = n_0 + 3k$, $k = 0, 1, 2, \ldots, |p| - 1$.

For u-$\mathrm{P}_{\mathrm{II}-2}$ multistep solutions exist for $a = 3p$ with integer p. When $n << 0$, the asymptotic behaviour of X is $X = a = 3p$, while for $n >> 0$ we find $X = p$. The multistep solution is the following: $X = 3p$ up to $n = 1 - 3|p|$ then we have $|p|$ times the elementary pattern of two jumps of $-p/|p|$ followed by two steps with constant value. However the last two steps of the last pattern are indistinguishable from the asymptotic value $X = p$ which is therefore obtained at point $|p| - 1$.

In the case of u-$\mathrm{P_{III}}$ we start by studying the asymptotic behaviours. At $n \to +\infty$, we find that the only simple behaviour of X is $X = 0$, while for $n \to -\infty$ one can have $X = \alpha n + \beta$. In order to simplify our search we will limit ourselves to solutions connecting a zero half-solution at $+\infty$ to a half-solution with *zero slope* (i.e. $\alpha = 0$) at $-\infty$. We find that the condition is that a and b be of the same parity, and the asymptotic value of X as $n \to -\infty$ is $X = |a - b|/2$. The multistep solution comprises $|a - b|/2$ jumps of one (one every three steps). Starting at large positive n, X is zero down to $n = \max(a, b) - 1$ then enters the multistep region and reaches the asymptotic value

$|a-b|/2$ at point $n = 1 - |a-b|/2 + \min(a,b)$.

For the higher ultra-discrete Painlevé equations we shall contend ourselves with exhibiting just a few simple solutions. Thus for u-P_{IV-1} we find that a solution $X = (-n)_+$, $Y = -2n - 1$, exists whenever $a = b$. For u-P_{IV-2}, the simplest solution exists for $a = 1$, $b = 0$. We have $X = (-n)_+$ and $Y = -n$ for $n \geq -1$, $Y = -2n - 1$ for $n \leq -1$. Finally, for u-P_V, a simple solution can be found for $a = b = 0$ and $c = 2k + 1$. In this case, $X = 0$ while $Y_n = 0$ for $n \leq -k - 1$, $Y_n = -n - k - 1$ for $-k - 1 \leq n \leq k$ and for $n \geq k$, $Y_n = -2n - 1$.

The investigation of the most general special solution for each u-Painlevé is still an open (and highly nontrivial) problem. Still, our analysis above has shown that the ultra-discrete Painlevé's have the right structure to possess rich particular solutions. As we have already pointed out in [], the existence of particular solutions can be used in order to fine-tune the form of the equation. As a matter of fact, unless the n-dependence of the u-Painlevé's is exactly the one given in equations (4.14) to (4.20), no family of particular solutions seems to exist.

6. The auto-Bäcklund transforms of the ultra-discrete Painlevé equations

The continuous and discrete Painlevé equations are characterised by a host of interrelations. These relations are of two kinds: either they connect the solution of one d-P to that of some other (Miura transformation) or they connect the solutions of a given d-P corresponding to different values of the parameters (auto-Bäcklund or Schlesinger transformations). As we will show, analogous relations exist for the ultra-discrete Painlevé equations.

In order to illustrate this, we shall work out in detail the auto-Bäcklund transformation for u-P_{II-1}. Let us start with the discrete P_{II}:

$$x_{n-1}x_{n+1} = \frac{\alpha z(x_n + z)}{x_n(1 + x_n)} \tag{6.1}$$

where $z = \lambda^n$. We readily remark that (6.1) is invariant under the transformation (I) $\alpha \to 1/\alpha$, $x \to z/x$. The Miura transformation (M) relates d-$\mathrm{P_{II}}$ to d-$\mathrm{P_{34}}$. It is given as a system:

$$y_n = x_n(x_{n+1} + 1) \tag{6.2}$$

$$x_n = \frac{y_n y_{n-1} - \alpha z^2}{y_{n-1} + \alpha z}$$

Eliminating y we obtain (6.1), while eliminating x we obtain d-$\mathrm{P_{34}}$ for y:

$$(y_n y_{n-1} - \alpha z^2)(y_n y_{n+1} - \alpha\lambda^2 z^2) = \alpha z(y_n + z)(y_n + \alpha\lambda z) \tag{6.3}$$

(The form (6.3) is not the canonical one and a gauge transformation $y \to z\sqrt{\alpha\lambda}y$ is needed in order to convert it to canonical form). Equation (6.3) is invariant under the transformation (J): $y \to \alpha\lambda y$, $\alpha \to 1/\alpha\lambda^2$. In order to obtain the auto-Bäcklund transformation for d-$\mathrm{P_{II}}$ (6.1) one must use the Miura to transform to d-$\mathrm{P_{34}}$, use the invariance of the latter and come back through the inverse Miura. The auto-Bäcklund (in fact, the Schlesinger) of d-$\mathrm{P_{II}}$ is thus $B = M^{-1}JMI$. It transforms the solution, x, of d-$\mathrm{P_{II}}$ with parameter α to one, $\tilde{x}$, corresponding to a parameter $\tilde{\alpha} = \alpha/\lambda^2$. Following the chain of transformation we find:

$$\tilde{x}_n = \frac{z(\lambda x_n x_{n-1} + \alpha(x_n + z))}{\lambda x_n(x_n x_{n-1} + x_n + z)} = \frac{\alpha z(\lambda z + x_{n+1}(x_n + 1))}{\lambda x_n(\alpha z + x_{n+1}(x_n + 1))} \tag{6.4}$$

where the expressions of $\tilde{x}$ are equivalent, as a consequence of (6.1). Using the chain $IM^{-1}JM$ one can compute the inverse Schlesinger leading to $\underset{\sim}{\alpha} = \alpha\lambda^2$.

The ultra-discrete limit of (6.1) is readily obtained:

$$X_{n+1} + X_{n-1} + X_n = a + 2n + (X_n - n)_+ - (X)_+ \tag{6.5}$$

(This is not in the canonical form we encountered in section 4, but can be easily transformed into it). The ultra-discretisation procedure cannot be applied to the Miura (6.2) and, in particular, to the second half of it. This means that in the ultra-discrete limit we can compute y_n from given x_n, x_{n+1} but the knowledge of y_n, y_{n-1} does not allow us to compute x_n. Still, the remarkable result is that, while the intermediate step is missing, the end result (6.4) is ultra-discretisable. Thus one can give the ultra-discrete form of the auto-Bäcklund:

$$\tilde{X}_n = n - 1 - X_n + \max(X_n + X_{n-1} + 1, a + X_n, a + n) - \max(X_n + X_{n-1}, X_n, n)$$
$$= a + n - 1 + \max(X_n + X_{n+1}, X_{n+1}, n + 1) - \max(X_n + X_{n+1}, X_{n+1}, a + n) \quad (6.6)\ cr$$

It is interesting to show that is X_n is a solution of (6.5) corresponding to a parameter a, then $\tilde{X}_n$ is a solution to (6.5) corresponding to parameter $a - 2$. This can be shown easily if one considers the asymptotic behaviour of the solutions of (6.5). We find that for $n >> 0$ the solution behaves like $X_n \sim n/2 + a/4$. Inserting this asymptotic solution into the auto-Bäcklund (6.6) we find that $\tilde{X}_n$ has a behaviour $\tilde{X}_n \sim n/2 + \tilde{a}/4$ where, precisely, $\tilde{a} = a - 2$.

In the case of d-$\mathrm{P_{III}}$ (4.6), the auto-Bäcklund transform has been derived in []. It reads:

$$\tilde{x}_n = \frac{x_n(\beta + \alpha\delta\lambda z x_{n-1}) + \alpha(\lambda x_n + z\beta)}{x_n^2\delta(\lambda + \gamma z x_{n-1}) + x_n(\alpha\delta\lambda z + \gamma x_{n-1})} = \frac{x_n(\lambda\alpha + \beta\gamma z x_{n+1}) + \beta(x_{n+1} + \alpha\lambda z)}{x_n^2\gamma(1 + \delta\lambda z x_{n+1}) + x_n(\beta\gamma z + \delta\lambda x_{n+1})} \quad (6.7)$$

where $\tilde{x}$ corresponds to an equation with parameters $\tilde{\alpha} = \alpha\sqrt{\lambda}$, $\tilde{\beta} = \beta/\sqrt{\lambda}$, $\tilde{\gamma} = \gamma/\sqrt{\lambda}$ and $\tilde{\delta} = \delta\sqrt{\lambda}$. The ultra-discrete form of (6.7) is straightforward:

$$\tilde{X}_n = \max(X_n + b, X_{n-1} + X_n + a + d + n + 1, X_n + a + 1, a + b + n)$$
$$- \max(2X_n + d + 1, X_{n-1} + 2X_n + c + d + n, X_n + a + d + n + 1, X_n + X_{n-1} + c) \quad (6.8)$$

or equivalently

$$\tilde{X}_n = \max(X_n + a + 1, X_{n+1} + X_n + b + c + n, X_{n+1} + b, a + b + n + 1)$$
$$- \max(2X_n + c, X_{n+1} + 2X_n + c + d + n + 1, X_n + b + c + n, X_n + X_{n+1} + d + 1) \quad (6.9)$$

In the case of d-$\mathrm{P_{IV-1}}$ (4.7) the auto-Bäcklund transformation reads:

$$\tilde{x}_n = \frac{1 + \alpha x_n}{\rho z y_{n-1}} \quad (6.10a)$$

$$\tilde{y}_n = \frac{1+\tilde{\alpha}\tilde{x}_n}{\beta z x_n} \qquad (6.10b)$$

where $z=\lambda^n$ and $\rho=\sqrt{\alpha\beta/\lambda}$. Eliminating x and y between (6.10) and (4.7) we find that the variables $\tilde{x}$ and $\tilde{y}$ satisfy a d-$\mathrm{P}_{\mathrm{IV}-1}$ of the same form as (4.7) at point $\tilde{z}=z\beta/\alpha$ and with parameters $\tilde{\alpha}=\rho$, $\tilde{\beta}=\rho z/\tilde{z}$. The inverse auto-Bäcklund can also be simply given:

$$x_n = \frac{1+\rho\tilde{x}_n}{\beta z\tilde{y}_n} \qquad (6.11a)$$

$$y_{n-1} = \frac{1+\alpha x_n}{\tilde{\beta}\tilde{z}\tilde{x}_n} \qquad (6.11b)$$

The ultra-discretization of the auto-Bäcklund is straightforward. We find thus from (6.10):

$$\tilde{X}_n = (X_n+a)_+ - r - n - Y_{n-1} \qquad (6.12a)$$

$$\tilde{Y}_n = (\tilde{a}+\tilde{X}_n)_+ - b - n - X_n \qquad (6.12b)$$

where $r=(a+b-1)/2$, and an analogous expression from (6.11).

In order to obtain the auto-Bäcklund of u-P_{V} we start with a convenient form of d-P_{V}:

$$x_n x_{n+1} = 1+y_n \qquad (6.13a)$$

$$y_n y_{n-1} = \frac{(x_n+\mu\kappa)(x_n+1/\mu\kappa)(x_n+\mu/\kappa)(x_n+\kappa/\mu)}{(1+x_n\lambda^n\rho)(1+x_n\lambda^n/\rho)} \qquad (6.13b)$$

The auto-Bäcklund transformation reads:

$$\tilde{x}_n = \frac{\mu x_n + x_{n+1}/\mu + \kappa + 1/\kappa}{y_n\lambda^{n-1/2}} \qquad (6.14a)$$

$$\tilde{y}_{n+1} = \frac{\mu\sqrt{\lambda}\tilde{x}_n + \tilde{x}_{n+1}/(\mu\sqrt{\lambda}) + \rho + 1/\rho}{x_n\lambda^n} \qquad (6.14b)$$

The $\tilde{x}$, $\tilde{y}$ variables satisfy a d-P_{V} at point $\tilde{z}=z\sqrt{\lambda}$ (i.e. the modified equation is defined at the points $n+1/2$) with parameters: $\tilde{\mu}=\mu\sqrt{\lambda}$, $\tilde{\kappa}=\rho$, $\tilde{\rho}=\kappa$. The ultradiscretization of (6.13) leads to an equation of the form (4.9) with $a=k+m$, $b=k-m$, $c=r$ where k,

m and r are the respective logarithms of κ, μ and ρ (times ϵ). The auto-Bäcklund in this case becomes:

$$\tilde{X}_n = \max(X_n + m, X_{n-1} - m, k, -k) - n + 1/2 - Y_n \tag{6.15a}$$

$$\tilde{Y}_{n+1} = \max(\tilde{X}_n + m + 1/2, \tilde{X}_{n+1} - m - 1/2, r, -r) - n - X_n \tag{6.15b}$$

¿From the above examples it becomes clear that the construction of the auto-Bäcklund transformation for u-Painlevé does not present fundamental difficulties. Once the auto-Bäcklund for the corresponding discrete equation is established, one can proceed to the construction of the auto-Bäcklund of the ultra-discrete equations. In fact, the procedure is limited only by the still fragmentary knowledge of the Miura/Bäcklund/Schlesinger of the discrete Painlevé equations.

7. Conclusion

In this paper, we have presented a systematic derivation of the ultra-discrete analogs of the Painlevé equations. Starting from the appropriate (multiplicative) discrete forms and applying the ultra-discretization procedure, we have obtained equations that extend the Painlevé transcendents to the domain of cellular automata. We have shown that these ultra-discrete equations possess several properties characteristic of the Painlevé equaitons. They organize themselves in coalescence cascades, they possess particular solutions globally defined and they have auto-Bäcklund transformations. An interesting question that remains open at this stage is whether the u-Painlevé's can be linearized like their continuous and discrete counterparts.

Another interesting point that will require a deeper investigation is that of an integrability criterion for ultra-discrete systems. In [], we have shown that a first requirement is that of linear stability. This has made possible to limit considerably the possible forms of u-Painlevé equations. In the present paper, we have seen, in our analysis of the Riccati equation, that it is also important that the "backward" evolution be defined. Still, the slow-growth criterion, coupled to the existence (and uniqueness?) of a preimage are not sufficiently strong in order to fix in an unambiguous way the form of an integrable ultra-discrete equation. In fact the only reliable method at our disposal to produce integrable ultra-discrete systems is to start from an integrable *dicrete* system (the singularity confinement integrability criterion is fully operative in this case) and apply systematically the ultra-discretization procedure.