CONSTRUCTING SOLUTIONS TO THE ULTRA-DISCRETE PAINLEVE EQUATIONS

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Abstract

We investigate the nature of particular solutions to the ultra-discrete Painlevé equations. We start by analysing the autonomous limit and show that the equations possess an explicit invariant which leads naturally to the ultra-discrete analog of elliptic functions. For the ultra-discrete Painlevé equations II and III we present special solutions reminiscent of the Casorati determinant ones which exist in the continuous and discrete cases. Finally we analyse the discrete $\mathrm{P_I}$ equation and show how it contains both the continuous and the ultra-discrete ones as particular limits.

1. Introduction

The study of integrable cellular automata (CA) has received a substantial boost recently with the introduction of the ultra-discretisation method which allows a *systematic* construction of CA's starting from a given discrete system. At the heart of the method lies the transformation which relates the variable of the discrete system x to that of the ultra-discrete X, through $x = e^{X/\epsilon}$. An essential requirement is that the variable of the discrete equation assume only *positive* values. In practice this means that the ultra-discretisation will isolate the positive solutions of the discrete system. In order to obtain a cellular-automaton one performs that limit $\epsilon \to 0$. The cornerstone of the procedure is the identity $\lim_{\epsilon\to 0} \epsilon \log(e^{a/\epsilon} + e^{b/\epsilon}) = \max(a, b)$. Thus, if a, b are integer, the result of the operation $\max(a, b)$ will give an integer value.

The ultra-discretisation approach has made possible the systematic derivation of the CA-analogs of a host of integrable evolution equations. In a recent work, we have presented the ultra-discrete forms of paradigmatic integrable systems: the Painlevé equations ($\mathcal{P}$). Our approach was based on the ultra-discretisation procedure applied to the known discrete forms of the Painlevé equations. In order to ensure the positivity requirement the discrete forms considered were the multiplicative ones i.e. the q-Painlevé equations. Thus, for example, we start from the three expressions of q-$\mathrm{P_I}$:

$$x_n^\sigma x_{n+1} x_{n-1} = z x_n + 1 \tag{1.1}$$

where $\sigma = 0, 1, 2$, $z = \lambda^n$. From (1.1), with $\lambda = e^{1/\epsilon}$, we obtain the ultra-discrete forms:

$$X_{n+1} + X_{n-1} + \sigma X_n = \max(0, X_n + n) \tag{1.2}$$

Ultra-discrete forms have been proposed for all the Painlevé equations. One important question that can be raised at this point is whether the ultra-discrete $\mathcal{P}$'s (u-$\mathcal{P}$) are indeed Painlevé equations. That latter have been proposed, initially in a continuous setting, as equations defining new transcendents, thus extending the special functions to the nonlinear domain. It is by now clear that the discrete $\mathcal{P}$'s also fulfill the basic requirements and can be considered as defining new functions (of the appropriate discrete variable). However, the same depth of analysis is far from being reached for ultra-discrete $\mathcal{P}$'s. Thus, it is important that the properties of these new equations be studied in detail. In this paper we shall examine the particular solutions of the u-$\mathcal{P}$'s II and III. In the discrete case, as well as the continuous one, the Painlevé equations are known to possess particular solutions for special values of their parameters. We shall show that the same holds true for the ultra-discrete $\mathcal{P}$'s and give the ultra-discrete analog of the Casorati determinant-type solutions which have been established for the continuous and discrete $\mathcal{P}$'s.

In section 2 we shall start with a simpler problem namely that of the autonomous limit of the ultra-discrete $\mathcal{P}$'s. In the continuous and discrete autonomous cases, the solutions are explicitly known to be elliptic functions. We study here the ultra-discrete equations and establish an integral of motion (which exists only in the autonomous limit). This integral would define the ultra-discrete analog of the elliptic functions. However, contrary to the continuous and discrete cases this does not seem possible. Thus we address directly the question of the ultra-discrete analog of elliptic functions and show how the latter can be systematically constructed. Section 3 is devoted to the special solutions of u-$\mathrm{P_{II}}$ and u-$\mathrm{P_{III}}$: these are the ultra-discrete analogs of the Casorati determinant rational solutions of the continuous and discrete $\mathcal{P}$'s. Finally, we examine the u-$\mathrm{P_I}$ equation. The Painlevé I equation does not have any particular solutions. Still, it is very interesting to study the discrete P_I equation and show how its solution contains the ones of the continuous as well as of the ultra-discrete P_I. The transition from one to the other is mediated by the sign of the parameter. By changing this parameter we can move from a strictly positive, cellular automaton-like solution to the typical P_I solution with poles on the negative real axis.

2. Autonomous ultra-discrete equations and elliptic functions

Before proceeding to the study of special solutions of the u-$\mathcal{P}$'s, let us start with a simpler example, that of their autonomous limits. It is well known that the autonomous limits of continuous and discrete $\mathcal{P}$'s are solved in terms of elliptic functions. In all these cases the second order equation can be integrated once and it turns out that the resulting integral is just a known addition formula for elliptic functions. We can thus wonder whether these properties cross over to the ultra-discrete case.

Let us start with the multiplicative autonomous form of d-P_I:

$$x_n^\sigma x_{n+1} x_{n-1} = \alpha x_n + 1 \tag{2.1}$$

It is easy to show that all these cases possess an invariant. Thus we have:
For $\sigma = 0$

$$x_n^2 + x_{n-1}^2 + a(x_n + x_{n-1}) + 1 = k x_n x_{n-1} \tag{2.2a}$$

For $\sigma = 1$

$$x_n^2 x_{n-1}^2 + a x_n x_{n-1}(x_n + x_{n-1}) + (a^2+1)(x_n + x_{n-1}) + a = k x_n x_{n-1} \tag{2.2b}$$

For $\sigma = 2$

$$x_n^2 x_{n-1}^2 + a(x_n + x_{n-1}) + 1 = k x_n x_{n-1} \tag{2.2c}$$

Starting with these expressions, it is very easy to construct the invariants for the ultra-discrete case. Let us work this out explicitly in the $\sigma = 2$ case. The ultra-discrete equation reads:

$$X_{n+1} + X_{n-1} + 2X_n = \max(X_n + A, 0) \tag{2.3}$$

and the ultra-discretization of the invariant (2.2c) leads to

$$K = \max(X_n + X_{n-1}, A - X_n, A - X_{n-1}, -X_n - X_{n-1}) \tag{2.4}$$

Let us now show that (2.4) is indeed a conserved quantity of (2.3). Equation (2.3) can be rewritten as:
i) $X_{n+1} = -X_{n-1} - X_n + A \qquad \text{if } X_n \geq -A$
ii) $X_{n+1} = -X_{n-1} - 2X_n \qquad \text{if } X_n \leq -A$ (2.5)
In the case i), we start with K and compute the right-hand side of (2.4) at the next step, i.e. $(X_{n-1}, X_n) \to (X_n, X_{n+1})$. Thus, starting from $\max(X_n + X_{n-1}, A - X_n, A - X_{n-1}, -X_n - X_{n-1})$ we obtain the expression $\max(A - X_{n-1}, A - X_n, X_n + X_{n-1}, X_{n-1} - A)$. This last expression will be equal to K provided that neither $-X_n - X_{n-1}$ nor $X_{n-1} - A$ is the largest one in its respective set, i.e. the max is in each case one of the three other terms which appear in both sets. And indeed, since $X \geq -A$, one has $-X_n - X_{n-1} \leq A - X_{n-1}$ and similarly $X_{n-1} - A \leq X_{n-1} + X_n$. Thus the last argument does not play any role and in this case the values of K at steps n and $n+1$ coincide. Similarly, in the $X_n \leq -A$ case, we have the two values of K at steps n and $n+1$: $\max(X_n + X_{n-1}, A - X_n, A - X_{n-1}, -X_n - X_{n-1})$ and $\max(-X_n - X_{n-1}, A - X_n, A + 2X_n + X_{n-1}, X_n + X_{n-1})$. We must show here that neither $A - X_{n-1}$

nor $A + 2X_n + X_{n-1}$ can be the max of its respective set. We have in fact (using $X_n \le -A$) that $A - X_{n-1} \le -X_n - X_{n-1}$ and $A + 2X_n + X_{n-1} \le X_n + X_{n-1}$. Thus K is determined as the maximum of the three remaining arguments and its value is conserved from step n to step $n+1$. This proves that K is indeed an invariant of (2.3).

Still, the existence of this invariant does not suffice in order to define the ultra-discrete analog of an elliptic function. It is easy to see that the iteration of (2.4) considered as an ultra-discrete equation does not define X_{n+1} in terms of X_n in a unique way (in particular when the maximal term is $A - X_n$). Thus, contrary to the continuous and discrete cases (2.4) cannot be used as such for the definition of the analog of the elliptic functions. One must use the full, three-point, mapping (or find a different approach to this question). Let us first show a typical solution of (2.3), Figure 1. The periodicity that characterizes the elliptic functions is clearly seen in the solution.

Since the invariant, considered as a ultra-discrete equation, cannot be used to define the ultra-discrete elliptic functions a more direct approach is needed. We must, as usually, go back to the discrete case and once the result is firmly established we can proceed to its ultra-discretisation. Let us start with the differential equation:

$$\frac{dx}{\sqrt{P}} + \frac{dy}{\sqrt{Q}} = 0, \tag{2.6}$$

where

$$P \equiv -(x-a)(x-b)(x-c), \quad Q \equiv -(y-a)(y-b)(y-c),$$

and a, b, c are constants (complex in general). A standard argument gives an integral of the differential equation (2.6) as

$$\frac{(\sqrt{P} - \sqrt{Q})^2}{(x-y)^2} = -(x+y) + C, \tag{2.7}$$

where C is an integration constant. If we choose $C = a + b + c$, eq. (2.7) is equivalent to the following algebraic equation:

$$x^2y^2 - 2\beta xy + \alpha(x+y) + \gamma = 0, \tag{2.8}$$

where $\alpha \equiv 4abc$, $\beta \equiv ab + bc + ca$ and $\gamma \equiv a^2b^2 + b^2c^2 + c^2a^2 - 2abc(a+b+c)$. This is exactly the integral we need for the autonomous dPI eq. In fact, if both $(x, y) = (x_{n+1}, x_n)$ and $(x, y) = (x_n, x_{n-1})$ satisfy eq. (2.8) and $x_{n+1} \ne x_{n-1}$, we can easily show

$$x_{n+1}x_n^2x_{n-1} = \alpha x_n + \gamma. \tag{2.9}$$

In order to take ultra-discrete limit, we confine ourselves to the case where $x_n > 0$ (for $\forall n$), $\alpha > 0$ and $\gamma > 0$.

In order to parametrize eq.(2.8). we assume

$$\begin{aligned} &a > b > c > 0, \\ &a > x > b, \quad a > y > b, \\ &a > \frac{bc}{(\sqrt{b} - \sqrt{c})^2}. \end{aligned} \tag{2.10}$$

These are sufficient conditions to the positivity we required. (It may be possible that we can obtain another u-limit when we choose different conditions.) Then, since (x, y) in eq.(2.8) are a solution to eq.(2.6), they satisfy

$$\int_x^a \frac{dx}{\sqrt{P}} + \int_y^a \frac{dy}{\sqrt{Q}} = \text{ const.}$$

Using the identity

$$\begin{aligned}\int_x^a \frac{dx}{\sqrt{P}} &= \frac{2}{\sqrt{a-c}}\mathrm{sn}^{-1}\left(\sqrt{\frac{a-x}{a-b}}; \sqrt{\frac{a-b}{a-c}}\right), \\ &\equiv \frac{2}{\sqrt{a-c}}u,\end{aligned} \tag{2.11}$$

we have a parametrization

$$\begin{aligned}x &= a-(a-b)\mathrm{sn}^2(u;k), \\ y &= a-(a-b)\mathrm{sn}^2(v;k),\end{aligned} \tag{2.12}$$

with $u+v=\xi$ (const.) and $k \equiv \sqrt{\dfrac{a-b}{a-c}}$. (Here $\mathrm{sn}(u;k)$, $\mathrm{cn}(u;k)$ and $\mathrm{dn}(u;k)$ are Jacobian elliptic functions and k is the modulus.)

The constant ξ is obtained as follows. Differentiating eqs. (2.11) and (2.12), we find

$$-\frac{1}{\sqrt{P}} = \frac{2}{\sqrt{a-c}}\frac{du}{dx},$$

$$\frac{dx}{du} = -2(a-b)\mathrm{sn}u\mathrm{cn}u\mathrm{dn}u.$$

Thus we get

$$\sqrt{P} = \sqrt{a-c}(a-b)\mathrm{sn}u\mathrm{cn}u\mathrm{dn}u. \tag{2.13}$$

Putting eqs. (2.12) and (2.13) into eq.(2.7) with $C=a+b+c$, we get

$$\begin{aligned}\left(\frac{\mathrm{sn}u\mathrm{cn}u\mathrm{dn}u-\mathrm{sn}v\mathrm{cn}v\mathrm{dn}v}{\mathrm{sn}^2v-\mathrm{sn}^2u}\right)^2 &= \frac{a-b}{a-c}(\mathrm{sn}^2u+\mathrm{sn}^2v)+\frac{b+c-a}{a-c}, \\ &= k^2(\mathrm{sn}^2u+\mathrm{sn}^2v)-1+\frac{b}{a-c}.\end{aligned}$$

Since we know $u+v=\xi$ (const.), taking $u=0$ and $v=\xi$, we obtain

$$\frac{\mathrm{dn}^2\xi}{\mathrm{sn}^2\xi} = \frac{b}{a-c},$$

or, equivalently,

$$\mathrm{sn}^2\xi = \mathrm{sn}^2(u+v) = \frac{a-c}{a}. \tag{2.14}$$

The above relation is also obtained from the identity

$$\left(\frac{\mathrm{sn}u\mathrm{cn}u\mathrm{dn}u-\mathrm{sn}v\mathrm{cn}v\mathrm{dn}v}{\mathrm{sn}^2v-\mathrm{sn}^2u}\right)^2 - k^2(\mathrm{sn}^2u+\mathrm{sn}^2v)+(1+k^2) = \frac{1}{\mathrm{sn}^2(u+v)},$$

which can be proved by the addition formulae of Jacobian elliptic functions. We should regard eq. (2.14) as definition of ξ.

Thus we obtain an elliptic solution to eq. (2.9). It is given as

$$x_n = f(u_0-n\xi), \tag{2.15}$$

where

$$f(u) \equiv a-(a-b)\mathrm{sn}^2(u;k) = a\mathrm{cn}^2(u;k)+b\mathrm{sn}^2(u;k),$$

and we use the fact $f(-u)=f(u)$.

The ultra-discretization of the autonomous dPI can be done in a straightforward way. To begin with, we rewrite sn and cn functions in terms of elliptic theta functions. The definitons of elliptic theta functions are

$$\vartheta_0(\nu) \equiv \sum_{n=-\infty}^{\infty} (-1)^n q^{n^2} z^{2n},$$

$$\vartheta_1(\nu) \equiv \sqrt{-1} \sum_{n=-\infty}^{\infty} (-1)^n q^{(n-1/2)^2} z^{2n-1},$$

$$\vartheta_2(\nu) \equiv \sum_{n=-\infty}^{\infty} q^{(n-1/2)^2} z^{2n-1},$$

$$\vartheta_3(\nu) \equiv \sum_{n=-\infty}^{\infty} q^{n^2} z^{2n},$$

where $z = \exp\left[\sqrt{-1}\pi\nu\right]$ and q is a complex constant (nome). We set

$$q = \exp\left[-\frac{\epsilon\pi}{\theta}\right]. \tag{2.16}$$

Using the Poisson's summation formulae, we get

$$\vartheta_0(\nu) = \sqrt{\frac{\theta}{\epsilon\pi}} \sum_{n=-\infty}^{\infty} \exp\left[-\frac{\theta}{\epsilon}[\nu-(n+1/2)]^2\right],$$

$$\vartheta_1(\nu) = \sqrt{\frac{\theta}{\epsilon\pi}} \sum_{n=-\infty}^{\infty} (-1)^n \exp\left[-\frac{\theta}{\epsilon}[\nu-(n+1/2)]^2\right],$$

$$\vartheta_2(\nu) = \sqrt{\frac{\theta}{\epsilon\pi}} \sum_{n=-\infty}^{\infty} (-1)^n \exp\left[-\frac{\theta}{\epsilon}[\nu-n]^2\right],$$

$$\vartheta_3(\nu) = \sqrt{\frac{\theta}{\epsilon\pi}} \sum_{n=-\infty}^{\infty} \exp\left[-\frac{\theta}{\epsilon}[\nu-n]^2\right].$$

The relations between Jacobian sn, cn functions and theta functions are given,

$$\mathrm{sn}u = \frac{\vartheta_3(0)\vartheta_1(\nu)}{\vartheta_2(0)\vartheta_0(\nu)}, \quad \mathrm{cn}u = \frac{\vartheta_0(0)\vartheta_2(\nu)}{\vartheta_2(0)\vartheta_0(\nu)},$$

with $u = \pi(\vartheta_3(0))^2\nu \equiv K\nu$ and $k^2 = \dfrac{a-b}{a-c} = \left(\dfrac{\vartheta_2(0)}{\vartheta_3(0)}\right)^4$.

We parametrize a, b, c as

$$c = \exp\left[-\frac{\eta\theta}{\epsilon}\right],$$

$$b = c + \left(\frac{\epsilon\pi}{\theta}\right)^2 (\vartheta_0(0))^4,$$

$$a = c + \left(\frac{\epsilon\pi}{\theta}\right)^2 (\vartheta_3(0))^4.$$

(This parametrization with (2.16) is not a unique one. There may be other parametrizations which lead to different u-limits.) Noticing the facts

$$\vartheta_0(0) \sim 2\sqrt{\frac{\theta}{\epsilon\pi}} \exp\left[-\frac{\theta}{4\epsilon}\right] \quad (\epsilon \to +0),$$

$$\vartheta_2(0) \sim \sqrt{\frac{\theta}{\epsilon\pi}} \left(1 - 2\exp\left[-\frac{\theta}{\epsilon}\right]\right) \quad (\epsilon \to +0),$$

$$\vartheta_3(0) \sim \sqrt{\frac{\theta}{\epsilon\pi}} \left(1 + 2\exp\left[-\frac{\theta}{\epsilon}\right]\right) \quad (\epsilon \to +0),$$

$$(\vartheta_3(\nu))^4 = (\vartheta_0(\nu))^4 + (\vartheta_2(\nu))^4 - (\vartheta_1(\nu))^4,$$

we find in the limit $\epsilon \to +0$,

$$a \sim 1,\ b \sim \exp\left[-\frac{\eta\theta}{\epsilon}\right],\ \ c \sim \exp\left[-\frac{\eta\theta}{\epsilon}\right],\ \ b-c \sim \exp\left[-\frac{\theta}{\epsilon}\right],$$
$$\alpha = 4abc \sim \exp\left[-\frac{2\eta\theta}{\epsilon}\right],$$
$$\gamma = (b-c)^2a^2 + ... \sim \exp\left[-\frac{2\theta}{\epsilon}\right] \quad .$$

Since a, b, c must satisfy the inequality (2.10) and

$$\frac{bc}{(\sqrt{b}-\sqrt{c})^2} = \frac{bc(\sqrt{b}+\sqrt{c})^2}{(b-c)^2} \sim \exp\left[-\frac{(3\eta-2)\theta}{\epsilon}\right] \quad (\epsilon \to +0),$$

we find the region of η as

$$\frac{2}{3} < \eta < 1.$$

We also find

$$(\vartheta_0(\nu))^2 \sim \left(\frac{\theta}{\pi\epsilon}\right)\left(\exp\left[-\frac{\theta}{\epsilon}[((\nu))-1/2]^2\right] + \exp\left[-\frac{\theta}{\epsilon}[((\nu))+1/2]^2\right]\right)^2,$$
$$(\vartheta_1(\nu))^2 \sim \left(\frac{\theta}{\pi\epsilon}\right)\left(\exp\left[-\frac{\theta}{\epsilon}[((\nu))-1/2]^2\right] - \exp\left[-\frac{\theta}{\epsilon}[((\nu))+1/2]^2\right]\right)^2,$$
$$(\vartheta_2(\nu))^2 \sim \left(\frac{\theta}{\pi\epsilon}\right)\left(\exp\left[-\frac{\theta}{\epsilon}[((\nu))]^2\right] - \exp\left[-\frac{\theta}{\epsilon}[((\nu))-1]^2\right]\right)^2,$$

where $((\nu)) \equiv \nu - \mathrm{Floor}(\nu)$, and $\mathrm{Floor}(x)$ is the maximum integer which does not exceeds x. Thus we get the asymptotic form of Jacobian elliptic functions as

$$\mathrm{sn}^2u \sim 1,\ \ \mathrm{cn}^2u \sim \left(\exp\left[-\frac{2\theta}{\epsilon}((\nu))\right] + \exp\left[-\frac{2\theta}{\epsilon}[1-((\nu))]\right]\right).$$

From eq.(2.14) or $\mathrm{cn}^2(u+v) = \dfrac{c}{a}$, we also get a relation for $\nu \equiv \dfrac{1}{K}u$ and $\nu' \equiv \dfrac{1}{K}v$ in the limit $\epsilon \to +0$,

$$\eta = 2\min\left[((\nu+\nu')),\ 1-((\nu+\nu'))\right].$$

Thus we find

$$\nu' = \pm\frac{\eta}{2} - \nu + \text{an arbitrary integer.}$$

Using the identity

$$\min[x,y] = \lim_{\epsilon\to+0} -\log\left[\exp\left(-\frac{x}{\epsilon}\right) + \exp\left(-\frac{y}{\epsilon}\right)\right],$$

we obtain the ultra discrete limit of eq.(2.9) as

$$X_{n+1} + 2X_n + X_{n-1} = \max[X_n + (\frac{3}{2} - 2\eta)\theta,\ 0]. \tag{2.17}$$

The elliptic solution of eq.(2.17) is obtained from eq. (2.15) using the asymptotic properties of elliptic functions as

$$X_n = \theta(\frac{1}{2} - \min[\eta, 2\nu_n, 2-2\nu_n]), \tag{2.18}$$

where

$$\nu_n = ((\nu_0 - n\frac{\eta}{2})) = \nu_0 - n\frac{\eta}{2} - \mathrm{Floor}(\nu_0 - n\frac{\eta}{2}).$$

It should be noted that if $x_n(\epsilon)$ is a solution to eq.(2.9) and the limit $\lim_{\epsilon\to+0}\log x_n(\epsilon)\equiv X_n-\theta/2$ exists, then X_n is a solution to eq.(2.17). This fact proves that (2.18) is a solution to eq.(2.17). (Note that eq. (2.18) is invariant under exchange of $\nu_0-n\frac{\eta}{2}$ by $-\nu_0+n\frac{\eta}{2}$.)

If we set $\theta=p$, $\eta=\frac{q}{p}$ and $\nu_0=\frac{r}{p}$ where p,q,r ($\frac{2}{3}<\frac{q}{p}<1$) are integers, we can regard eq.(2.17) as an evolution equation which takes values only in integers. It is also easy to see that X_n is periodic with period at most $2p$. Choosing θ, η and ν_0 we can show that (2.18) reproduces exactly the solution displayed in Figure 1. Thus we have explicitly constructed the elliptic function solutions to the autonomous u-$\mathrm{P_I}$ (2.4). In a similar way we can proceed to the construction of the solutions of the other autonomous forms.

3. Special solutions of ultra-discrete $\mathrm{P_{II}}$ and $\mathrm{P_{III}}$

It is well known that the continuous and discrete $\mathcal{P}$'s can be transformed into bilinear forms involving τ functions. Each of the latter is an entire function associated with the singularity of the $\mathcal{P}$'s, that is, the singularities are produced by the zeros of τ functions appearing in the denominator of the nonlinear variable of $\mathcal{P}$'s. However in the ultra-discrete case, the analogs of the notions of singularity and entire function have not yet been clearly defined. Then what is the τ function in the ultra-discrete world? In a naïve sense, an entire function does not have any factor in the denominator. Therefore it is naturally expected that the τ function of ultra-discrete case has no term of negative sign and can be represented in as a sum of only positive terms. This is due to the fact that the ultra-discrete limit "$\lim_{\epsilon\to0}\epsilon\log$" transforms the factors in the numerator into the positive terms and those in the denominator into the negative ones. Let us consider the simplest case, the rational solutions of $\mathcal{P}$'s. For the continuous $\mathcal{P}$'s, the τ functions of rational solutions are expressed by a Wronskian determinant with simple polynomial components. Also for discrete $\mathcal{P}$'s, it has been shown (at least for the cases analyzed up to now) that the τ functions of rational solutions are given in the form of a Casorati determinant, the discrete analog of the Wronskian. In both cases, the bilinear equations of τ's are directly obtained by using a simple determinant identity, the so-called Plücker relation. We can thus expect that the u-$\mathcal{P}$'s are transformed into an ultra-discrete bilinear form in terms of τ functions which are expressible as a termwise positive sum, and in that expression the derivation of the bilinear equations becomes transparent.

In the following, we give the rational solutions of u-$\mathrm{P_{II}}$ and u-$\mathrm{P_{III}}$ which are represented in a form reminiscent of the ultra-discrete analogs of the Casorati determinant.

Let us start with the u-$\mathrm{P_{II}}$ of first kind,

$$X_{n+1}+X_{n-1}=\max(0,n-X_n)-\max(0,X_n+n-a), \tag{3.1}$$

which is derived from the multiplicative d-$\mathrm{P_{II}}$,

$$x_{n+1}x_{n-1}=\frac{x_n+z}{x_n(1+\alpha z x_n)},\qquad z=\lambda^n,\quad \alpha\text{: parameter}, \tag{3.2}$$

by replacing $x=e^{X/\epsilon}$, $\lambda=e^{1/\epsilon}$, $\alpha=e^{-a/\epsilon}$ and taking the limit $\epsilon\to0$. Through the variable transformation,

$$X_n=\tau_{n-1}-\tau_{n-2}, \tag{3.3}$$

we obtain the ultra-discrete "bilinear" equation,

$$\tau_n+\max(\tau_{n-2},\tau_{n-1}+n-a)=\tau_{n-3}+\max(\tau_{n-1},\tau_{n-2}+n). \tag{3.4}$$

We remark that in the ultra-discrete world, "max" and "+" should be regarded as "addition" and "multiplication" respectively, because $e^{A/\epsilon}+e^{B/\epsilon}$ and $e^{A/\epsilon}e^{B/\epsilon}$ go to $\max(A,B)$ and $A+B$ respectively under the operation of $\lim_{\epsilon\to 0}\epsilon\log$. Thus the above equation is indeed the bilinear form in τ. Now (3.1) is invariant under the transformation $a\to -a$, $n\to n-a$, $X\to -X$, thus we can assume $a\geq 0$ without loss of generality. The u-$\mathrm{P_{II}}$ (3.1) admits rational solutions for $a=4m$, m: non-negative integer. The τ function for the rational solution is given as

$$\tau_n=\sum_{j=0}^{m-1}\max(0,n-3j), \tag{3.5}$$

which is also expressed as

$$\tau_n=\max_{0\leq j\leq m}\big(jn-\frac{3}{2}j(j-1)\big). \tag{3.6}$$

The existence of the above two expressions is essential in the proof of the bilinear equations as we will see below. This τ_n gives the m-step solution X_n (3.3) which have m successive jumps of step 1 at $n=3j-1$, $1\leq j\leq m$.

Let us consider a slightly more general form of τ function,

$$\tau_n=\sum_{j=0}^{m-1}\max(0,n-jk), \tag{3.7a}$$

$$=\max_{0\leq j\leq m}\big(jn-\frac{j(j-1)}{2}k\big), \tag{3.7b}$$

where k is positive. For positive p and q, we have

$$\begin{aligned}&\max\big(\tau_n,\tau_{n+p-k}+n-(m-1)p-q\big)\\&=\max\Big(\max_{0\leq j\leq m}\big(jn-\frac{j(j-1)}{2}k\big),\max_{1\leq j\leq m+1}\big(jn-\frac{j(j-1)}{2}k+(j-m)p-q\big)\Big)\end{aligned}$$

in the rhs, the j-th term in the second max is less than the j-th term in the first max for $1\leq j\leq m$, thus we can drop these terms,

$$\begin{aligned}&=\max\Big(\max_{0\leq j\leq m}\big(jn-\frac{j(j-1)}{2}k\big),(m+1)n-\frac{m(m+1)}{2}k+p-q\Big)\\&=\max\big(0,n,2n-k,\cdots,mn-\frac{m(m-1)}{2}k,(m+1)n-\frac{m(m+1)}{2}k+p-q\big)\end{aligned}$$

and taking $p-q\leq k$, we get

$$=\max(0,n)+\max(0,n-k)+\cdots+\max(0,n-(m-1)k)+\max(0,n-mk+p-q),$$

because the second arguments of the above maxima are numbers in decreasing order. Hence we obtain

$$\max\big(\tau_n,\tau_{n+p-k}+n-(m-1)p-q\big)=\tau_n+\max(0,n-mk+p-q),\quad p\geq 0,q\geq 0,p-q\leq k. \tag{3.8}$$

Similarly we have

$$\begin{aligned}&\max(\tau_n,\tau_{n+p-k}+n+q)\\&=\max\Big(\max_{0\leq j\leq m}\big(jn-\frac{j(j-1)}{2}k\big),\max_{1\leq j\leq m+1}\big(jn-\frac{j(j-1)}{2}k+(j-1)p+q\big)\Big)\end{aligned}$$

where the j-th term in the second max is dominant for $1\leq j\leq m$, so

$$\begin{aligned}&=\max\Big(0,\max_{1\leq j\leq m+1}\big(jn-\frac{j(j-1)}{2}k+(j-1)p+q\big)\Big)\\&=\max\big(0,n+q,2n+q+p-k,\cdots,(m+1)n+q+mp-\frac{m(m+1)}{2}k\big)\end{aligned}$$

again taking $p-q\leq k$, we get

$$= \max(0, n+q) + \max(0, n+p-k) + \max(0, n+p-2k) + \cdots + \max(0, n+p-mk).$$

Thus we obtain

$$\max(\tau_n, \tau_{n+p-k} + n + q) = \max(0, n+q) + \tau_{n+p-k}, \quad p \geq 0, q \geq 0, p-q \leq k. \tag{3.9}$$

From (3.8) through replacing $p \to p+q$, $n \to n-p$, and (3.9) through replacing $n \to n-r$, $p \to k+r-p$, $q \to r$, we get the bilinear equation,

$$\tau_n + \max\bigl(\tau_{n-p}, \tau_{n+q-k} + n - m(p+q)\bigr) = \tau_{n-k} + \max(\tau_{n-r}, \tau_{n-p} + n), \quad 0 \leq p \leq k, q \geq 0, r \geq 0, \tag{3.10}$$

where we have used $\tau_n - \tau_{n-k} = \max(0, n) - \max(0, n - mk)$ which follows from the explicit expression of τ (3.7a).

For $k = 3$, $p = 2$, $q = 2$ and $r = 1$, (3.10) reduces to (3.4), thus we have proved that (3.3) and (3.5) give the solution of u-P_{II} (3.1) with $a = 4m$. The remark is that expression (3.7a) looks similar to a Casorati determinant and (3.7b) to the expansion of the determinant. Let us recall the Casorati determinant representation of m-soliton solution for the discrete KP hierarchy. The determinant τ consists of the products of m components and each component of the determinant is the sum of two terms. Expanding the determinant, we get the expression of the sum of exponential terms. This situation is parallel to (3.7a) and (3.7b) except for a missing "max" in front of the summation in (3.7a).

Next we will consider the special solution for the u-P_{II} of second kind,

$$X_{n+1} + X_{n-1} - X_n = \max(0, n - X_n) - \max(0, X_n + n - a). \tag{3.11}$$

This is derived from another multiplicative d-P_{II},

$$x_{n+1}x_{n-1} = \frac{x_n + z}{1 + \alpha z x_n}, \qquad z = \lambda^n, \quad \alpha\text{: parameter}, \tag{3.12}$$

through $x = e^{X/\epsilon}$, $\lambda = e^{1/\epsilon}$, $\alpha = e^{-a/\epsilon}$ and $\epsilon \to 0$. The ultra-discrete bilinear form of (3.11) is

$$\tau_n + \max(\tau_{n-3}, \tau_{n-1} + n - a) = \tau_{n-4} + \max(\tau_{n-1}, \tau_{n-3} + n), \tag{3.13}$$

where $X_n = \tau_{n-1} - \tau_{n-3}$. For $a = 6m$, m: integer, there exist rational solutions of (3.12). The τ function for the solution is given by

$$\tau_n = \sum_{j=0}^{m-1} \max(0, n - 4j), \tag{3.14}$$

and the bilinear equation (3.13) is just the consequence of (3.10) with $l = 4$, $p = 3$, $q = 3$ and $r = 1$. For this τ function, X_n gives a multistep solution with the elementary pattern of two successive jumps at $n = 4j - 2$ $(1 \leq j \leq m)$ followed by two steps with constant value.

Let us proceed to the rational solution of u-P_{III}. We consider only a degenerate case, namely, the case in which the u-P_{III} is decomposed into two parts. The u-P_{III},

$$\begin{aligned} X_{n+1} + X_{n-1} - 2X_n = {} & \max(0, n - X_n) - \max(0, X_n + n - a) \\ & + \max(0, n - X_n + b) - \max(0, X_n + n + b), \end{aligned} \tag{3.15}$$

is derived from the d-P_{III},

$$x_{n+1}x_{n-1} = \frac{(x_n + z)(x_n + \beta z)}{(1 + \alpha z x_n)(1 + \beta z x_n)}, \qquad z = \lambda^n, \quad \alpha, \beta\text{: parameter}, \tag{3.16}$$

through $x = e^{X/\epsilon}$, $\lambda = e^{1/\epsilon}$, $\alpha = e^{-a/\epsilon}$, $\beta = e^{b/\epsilon}$ and $\epsilon \to 0$. Now decomposing (3.15) in the following two equations,

$$X_{n+1} + X_{n-1} = \max(0, n - X_n) - \max(0, X_n + n - a), \tag{3.17a}$$

$$2X_n = \max(0, X_n + n + b) - \max(0, n - X_n + b), \tag{3.17b}$$

we get the bilinear equations through $X_n = \tau_{n-1} - \tau_{n-2}$,

$$\tau_n + \max(\tau_{n-2}, \tau_{n-1} + n - a) = \tau_{n-3} + \max(\tau_{n-1}, \tau_{n-2} + n), \tag{3.18a}$$

$$\tau_{n-1} + \max(\tau_{n-1}, \tau_{n-2} + n + b) = \tau_{n-2} + \max(\tau_{n-2}, \tau_{n-1} + n + b). \tag{3.18b}$$

The first equation (3.17a) or (3.18a) is nothing but the u-P_{II} of first kind, therefore what we have to do is to prove that the solution (3.5) simultaneously satisfies the second bilinear equation (3.18b). ¿From (3.9) we obtain another bilinear equation,

$$\max(\tau_{n-q_1}, \tau_{n-p_1} + n) - \tau_{n-p_1} = \max(\tau_{n-q_2}, \tau_{n-p_2} + n) - \tau_{n-p_2}, \quad 0 \le p_i \le q_i + l, q_i \ge 0, \tag{3.19}$$

which gives (3.18b) by taking $p_1 = q_2 = b + 1$, $p_2 = q_1 = b + 2$ and $l = 3$. Hence we have proved that the same τ as in the rational solution of u-P_{II} gives the solution for the u-P_{III} (3.15) with $a = 4m$ (m: integer) and $b \ge -1$.

Solutions to the higher u-$\mathcal{P}$'s could be obtained following the above techniques (obviously with considerable technical difficulties).

4. Solutions of the P_I equation: continuous, discrete and ultra-discrete cases

In the previous sections, we have shown that, in perfect parallel to the continuous and discrete cases, the ultra-discrete P_{II} and P_{III} equations do possess special solutions. Thus, naturally, the question arises of how does the solution of the ultra-discrete equation relate to the solution of its continuous homolog. In order to perform this comparison we have chosen to analyze the P_I equation. The reason is that this equation does not have any special solutions and thus an arbitrary choice of initial conditions is expected to yield the generic behaviour of the solution. Our argument is (and has always been) that the *discrete* equation captures the essence of the behaviour of both its limiting cases, be it continuous or ultra-discrete. In order to show this explicitely we shall consider a specific example. We start from the discrete form of P_I: $x_{n-1}x_{n+1} = zx_n + 1$ where $z = a\lambda^n$. We introduce a scaling of x and z so as to rewrite the equation as:

$$x_{n-1}x_{n+1} = zx_n + b \tag{4.1}$$

We can, without loss of generality, assume that $a > 0$ (If $a < 0$ it suffices to change the sign of both x and z). Let us first look at the ultra-discrete case. For this we must assume that $b > 0$, and chosing initial conditions x_{n-1},$x_n > 0$ we find $x_{n+1} > 0$. Thus we can take the logarithm of x and introduce the new variable $X = \epsilon \log x$ where $\epsilon = 1/\log\lambda$. Figure (2) shows a typical behaviour of the solution of (4.1) for positive b where we have plotted the variable $\log(x_n)$ as a function of n.

Clearly this behaviour is only vaguely reminiscent of that of the continuous P_I: only the growing oscillating part for positive n resembles the one of the solution of the latter. The double poles present in the solution of P_I are absent.

The ultra-discrete limit corresponds to $\lambda \to \infty$ (or $\epsilon \to 0$) and lead to the equation

$$X_{n-1} + X_{n+1} = \max(X_n + n + A, B) \tag{4.2}$$

Going to the limit does not change the overall appearence of the solution of (4.2) as compared to that of (4.1) we presented in Fig 2. Simply, the values of X are now integers provided we start with integer values for X_{n-1}, X_n, A and B.

In order to obtain the continuous limit of (4.1), we introduce the following transformations: $x = \beta(1+\epsilon^2 w)$ and $z = 2\beta(1+\epsilon^4 t)$ with $b = -\beta^2$. The essential observation here is that for the continuous limit to exist we must have $b < 0$. In the limit $\epsilon \to 0$, we obtain the continuous $\mathrm{P_I}$ in the form:

$$w'' + w^2 - 2t = 0 \tag{4.3}$$

In Figure 3 we plot the solution of (4.1) for $b < 0$ where we represent $(x/\beta - 1)$ as a function of n. The appearance of poles is now clear.

These two simulations confirm our statement that the solution of the discrete equation contains the full richness of behaviour. In the particular example studied here, depending on the sign of b, we find ourselves either in the ultra-discrete or the continuous domain of behaviour. The precise limits are not essential: as soon as the sign of b changes, the qualitative changes in the solutions set in.

Finally, let us for the sake of completeness study the case $b = 0$. In this case (4.1) becomes:

$$x_{n-1}x_{n+1} = zx_n \tag{4.4}$$

and introducing again $X = \epsilon \log x$ we have the linear equation:

$$X_{n-1} + X_{n+1} = X_n + n \tag{4.5}$$

The solutions of (4.5) is straightforward: $X = n + \mu j^n + \nu j^{2n}$ where $j = e^{i\pi/3}$, i.e. a linear growth with a superimposed oscillating pattern of period 6.

5. Conclusion

In the preceding sections, we have studied the special solutions of the ultra-discrete Painlevé equations. We have shown that the analog of the bilinear formalism exists also in the ultra-discrete case and that it can be used in order to construct the ultra-discrete analog of the Casorati determinant solutions of the Painlevé equations. The present work is a first exploratory investigation in this direction: a complete study of the special solutions of the u-$\mathcal{P}$'s will necessitate a considerable effort (as a matter of fact, this program is not yet complete even in the continuous and discrete cases).

The autonomous limit of the ultra-discrete Painlevé equations has also been considered. In analogy to the continuous and discrete cases we expected the equations to possess an explicit invariant and be solve in terms of elliptic functions. We have shown that these statements are true and we have explicitly constructed the ultra-discrete analog of elliptic functions.

Finally our work has (hopefully) helped to illustrate the fundamental character of the discrete equations. Choosing the specific example of $\mathrm{P_I}$, we have shown by detailed numerical simulations that the discrete $\mathrm{P_I}$ can exhibit behaviours reminiscent of both the continuous and ultra-discrete $\mathrm{P_I}$ depending on the sign of a parameter. This is a further indication that the discrete equations are the fundamental entities of the integrable world.