CELLULAR AUTOMATA AND ULTRA-DISCRETE PAINLEVÉ EQUATIONS

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Abstract

Starting from integrable cellular automata we present a novel form of Painlevé equations. These equations are discrete in both the independent variable and the dependent one. We show that they capture the essence of the behavior of the Painlevé equations, organize themselves into a coalescence cascade and possess special solutions. A necessary condition for the integrability of cellular automata is also presented. We conclude with a discussion of the notion of integrability of the cellular automata under examination.

A novel extension of the Painlevé equations, well-known for their numerous applications in mathematics and physics, will be presented in this paper. Recent progress in the domain of integrable discrete systems has led to the derivation of discrete analogs of the Painlevé equations [1]. The discrete Painlevé equations are non-autonomous integrable mappings which, at the continuous limit, go over to the well-known differential equations of Painlevé. Discrete Painlevé equations were first identified in a 2D model of quantum gravity where they appeared as an integrable recursion relation for the calculation of the partition function [2]. With hindsight, their first occurence can be traced back to the work of Shohat on orthogonal polynomials [3]. To date, the full list of discrete Painlevé equations has been established [4] and their properties (in perfect parallel with those of their continuous counterparts) are being actively investigated. The notion of discrete integrability cannot be formulated in a unique, universally accepted way, a fact equally true for the continuous integrability. In [5] we have presented a classification of the various aspects of discrete integrability, which can be used as a *working* definition. Thus a discrete system is integrable if it possesses a sufficient number of first integrals or can be reduced to a linear system or if it possesses a Lax pair.

The study of discrete integrable systems has made clear the fact that they, rather than the continuous ones, are the fundamental entities. In fact, continuous systems are contained, through appropriate limits, in the discrete ones. A recent discovery has made possible to investigate the 'opposite' limit, that of ultra-discrete systems. A systematic way for the introduction of integrable ultra-discrete systems was proposed in [6]. In previous works on discrete systems, while the independent variables were discrete, the *dependent* variables were assumed to vary continuously. The ultra-discrete limit provides a systematic way to discretize the dependent variable also. One can, starting from a given evolution equation, obtain the cellular automaton (CA) equivalent. The aim of this paper is two-fold. First, we introduce the ultra-discrete analogs of the Painlevé equations and investigate their properties and, second, we provide integrability conditions for cellular-automaton like equations. Let us make clear from the outset what we mean by ultra-discrete Painlevé equations. They are the ultra-discrete limits of discrete Painlevé equations (the integrability of which is established by a discrete integrability criterion [7]). These limits are systematically obtained following the procedure of [6] and which will be summarized in what follows.

Cellular automata have been the object of an impressive number of studies and their behavior is known to be of the utmost richness. The integrability of such systems has not been thoroughly studied, since it represents considerable difficulties. An occurence of an integrable automaton has been noted in [8] by Pomeau who obtained explicitly its constant of motion. Cellular automata representing evolution equations have been studied from the point of view of the existence of localized, soliton-like, solutions. The notion of soliton for CA's was first introduced by Park et al. in [9]. Further examples of such CA's with soliton-like structures were given by the Clarkson group [10]. Integrable CA were introduced by Bruschi and collaborators [11] who derived Lax pairs for their cellular-automaton equations. Bobenko et al. [12] have proposed an interesting approach to integrable CA's by considering them as the restriction of an integrable discrete equation over a finite field. However in many cases the relation to the well-known integrable evolution equations was based on circumstancial evidence rather than a systematic derivation. The situation has changed recently due to the introduction of a method [6] that allows one to convert a given *discrete* evolution equation to one where the dependent variable also takes discrete values. The starting point was the CA proposed by one of the authors in collaboration with Satsuma [13]. This simple model (essentially a "box and ball" system) was shown later to be the ultra-discrete limit of the KdV equation [6]. This was obtained through a limiting procedure (the ultra-discrete limit) which allows one to derive a cellular automaton equation starting from the appropriate form of a discrete evolution equation.

As an illustration of the method and a natural introduction to ultra-discrete Painlevé equations, let us consider the following discrete Toda system introduced in [14]:

$$u_n^{t+1} - 2u_n^t + u_n^{t-1} = \log(1+\delta^2(e^{u_{n+1}^t} - 1)) - 2\log(1+\delta^2(e^{u_n^t} - 1)) + \log(1+\delta^2(e^{u_{n-1}^t} - 1)) \tag{1}$$

which is the integrable discretization of the continuous Toda system:

$$\frac{d^2 r_n}{dt^2} = e^{r_{n+1}} - 2e^{r_n} + e^{r_{n-1}} \tag{2}$$

For the ultra-discrete limit one introduces w through $\delta = e^{-L/2\epsilon}$, $w_n^t = \epsilon u_n^t - L$ and takes the limit $\epsilon \to 0$. The well-known result $\lim_{\epsilon \to 0} \epsilon \log(1 + e^{x/\epsilon}) = \max(0, x) = (x + |x|)/2 \equiv (x)_+$ must be used. The function $(x)_+$ is also known under the name of truncated power function and is equal to 0 for $x \leq 0$ and x for $x > 0$. Thus the ultra discrete limit of (1) becomes simply [15]:

$$w_n^{t+1} - 2w_n^t + w_n^{t-1} = (w_{n+1}^t)_+ - 2(w_n^t)_+ + (w_{n-1}^t)_+ \tag{3}$$

Since the truncated power function of any integer is a non-negative integer, it is clear that if w has initially integer values the values will remain integer at all subsequent time steps. Thus equation (3) is indeed a cellular automaton equation.

Let us now restrict ourselves to a simple periodic case with period two i.e. $r_{n+2} = r_n$ and similarly $w_{n+2} = w_n$. Calling $r_0 = x$ and $r_1 = y$ we have from (2) the equation $\ddot{x} = 2e^y - 2e^x$ and $\ddot{y} = 2e^x - 2e^y$ resulting to $\ddot{x} + \ddot{y} = 0$. Thus $x + y = \mu t + \nu$ and we obtain after some elementary manipulations:

$$\ddot{x} = ae^{\mu t}e^{-x} - 2e^x \tag{4}$$

Equation (4) is a special form of the Painlevé $\mathrm{P_{III}}$ equation. Indeed, putting $v = e^{x - \mu t/2}$, we find:

$$\ddot{v} = \frac{\dot{v}^2}{v} + e^{\mu t/2}(a - 2v^2) \tag{5}$$

The same periodic reduction can be performed on the ultra-discrete Toda equation (3). We introduce $w_0^t = X^t$, $w_1^t = Y^t$ and have, in perfect analogy to the continuous case, $X^{t+1} - 2X^t + X^{t-1} = 2(Y^t)_+ - 2(X^t)_+$ and $Y^{t+1} - 2Y^t + Y^{t-1} = 2(X^t)_+ - 2(Y^t)_+$. Again, $\Delta_t^2(X^t + Y^t) = 0$ and we can take $X^t + Y^t = mt + p$ (where m, t, p take integer values). We find thus that X obeys the ultra-discrete equation:

$$X^{t+1} - 2X^t + X^{t-1} = 2(mt + p - X^t)_+ - 2(X^t)_+ \tag{6}$$

This is the ultra-discrete analog of the special form (5) of the Painlevé $\mathrm{P_{III}}$ equation. Figure 1 gives a comparison of the evolution under (4) and (6). We remark that the dynamics are very similar: the two particles converge towards each other, rebound once or twice, get captured and go on oscillating around some equilibrium point. Thus, starting from a well-known physical problem we have introduced the corresponding cellular automaton equation and, restricting it to the simplest periodic lattice, we obtained the ultra-discrete form of a Painlevé equation.

In order to construct the ultra-discrete analogs of the Painlevé equations (u-P) we must start with the discrete form that allows the ultra-discrete limit to be taken. The general procedure is to start with an equation for x, introduce X through $x = e^{X/\epsilon}$ and then take appropriately the limit $\epsilon \to 0$. Clearly the substitution $x = e^{X/\epsilon}$ requires x to be positive. This is a stringent requirement that limits the exploitable form of the d-P's to multiplicative ones. Fortunately many such forms are known for the discrete Painlevé transcendents [16]:

d-$\mathrm{P_I}$

$$x_{n+1}x_{n-1} = \frac{\alpha\lambda^n}{x_n} + \frac{1}{x_n^2} \tag{7a}$$

$$x_{n+1}x_{n-1} = \alpha\lambda^n + \frac{1}{x_n} \tag{7b}$$

$$x_{n+1}x_{n-1} = \alpha\lambda^n x_n + 1 \tag{7c}$$

d-$\mathrm{P_{II}}$

$$x_{n+1}x_{n-1} = \frac{\lambda^n(x_n + \alpha\lambda^n)}{x_n(1 + x_n)} \tag{7d}$$

$$x_{n+1}x_{n-1} = \frac{x_n + \alpha\lambda^n}{1 + \beta x_n\lambda^n} \tag{7e}$$

d-$\mathrm{P_{III}}$

$$x_{n+1}x_{n-1} = \frac{(x_n + \alpha\lambda^n)(x_n + \beta\lambda^n)}{(1 + \gamma x_n\lambda^n)(1 + \delta x_n\lambda^n)} \tag{7f}$$

We remark that in some cases, more than one form exists for a given d-P. The derivation of the equivalent ultra-discrete forms is elementary: we introduce $\lambda = e^{1/\epsilon}$, take the logarithm of both sides of the equation and whenever a sum appears we apply the limit leading to the truncated power function. We find thus:

u-$\mathrm{P_I}$

$$X_{n+1} + X_{n-1} + 2X_n = (X_n + n + a)_+ \tag{8a}$$

$$X_{n+1} + X_{n-1} + X_n = (X_n + n + a)_+ \tag{8b}$$

$$X_{n+1} + X_{n-1} = (X_n + n + a)_+ \tag{8c}$$

u-$\mathrm{P_{II}}$

$$X_{n+1} + X_{n-1} = n + (n + a - X_n)_+ - (X_n)_+ \tag{8d}$$

$$X_{n+1} + X_{n-1} - X_n = (n + a - X_n)_+ - (X_n + n + b)_+ \tag{8e}$$

u-$\mathrm{P_{III}}$

$$X_{n+1}+X_{n-1}-2X_n = (n+a-X_n)_+ + (n+b-X_n)_+ - (X_n+c+n)_+ - (X_n+d+n)_+ \tag{8f}$$

These equations describe celular automata provided we restrict the values of the parameters as well as the initial values of the dependent variable to integers. Note that equation (6), for $m = 2$ is the subcase $a = b = p$, $c = d = 0$ of (8f) after the change of variable $X^t = X_n + n$, $(n \equiv t)$.

At this point two questions appear unavoidable. First, is it justified to call these equations ultra-discrete *Painlevé* equations? This is a question we shall address in detail in our conclusion. Meanwhile we shall show that the ultra-discrete equations we derived have some properties in common with the familiar Painlevé equations. One first remark is that the u-P's form a coalescence cascade just like their continuous and discrete counterparts [16]. Indeed, starting from u-$\mathrm{P_{III}}$ (8f) we can obtain u-$\mathrm{P_{II}}$ (8d) by taking $b \to +\infty$ and $c \to +\infty$ such that $b-c$ is finite. Next we translate X and through a linear transformation of n we find u-$\mathrm{P_{II}}$ (8d). The use of the identity $(x)_+ = x + (-x)_+$ is also necessary. Similarly, starting from u-$\mathrm{P_{II}}$ (8d) we can put $n = -2m - \rho$, $X = -m - Y$, $a = 2\rho$ and recover u-$\mathrm{P_I}$ (8a), for Y_m, at the limit $\rho \to \infty$. From u-$\mathrm{P_{II}}$ (8e) we obtain u-$\mathrm{P_I}$ (8b) simply by putting $X = Y - \rho$, $a = -\rho$ and $b = 2\rho$. We can also recover (8c) from (8e) and (8b) also from (8d). This is not the only property the u-P's share with the continuous and discrete Painlevé equations as we shall see below.

The second question is whether it is possible to guess the forms of the u-P's. In other words, what is the (integrability) criterion that would single them out among all possible equations? In the case of discrete systems the criterion for integrability (equivalent to the Painlevé property) is the property known as singularity confinement [7]. For cellular automata no singularity can exist and thus this criterion is inoperable. The situation is analogous to polynomial mappings where no singularity can appear. There, the criterion for integrability is based on arguments of growth of the degree of the iterate (or the similar notion of complexity introduced by Arnold [17]). Veselov [18] has shown that among mappings of the form $x_{n+1} - 2x_n + x_{n-1} = P(x_n)$ with polynomial $P(x)$, only the linear one has non-exponential growth of the degree of the polynomial that results from the iteration of the initial conditions. Let us apply such a low-growth criterion to a family of ultra-discrete $\mathrm{P_I}$ equations of the form:

$$X_{n+1} + \sigma X_n + X_{n-1} = (X_n + \phi(n))_+ \tag{9}$$

The three u-P_I obtained from (7) correspond to $\sigma = 0, 1, 2$. What is the condition for X not to grow exponentially towards $\pm\infty$? We ask simply that the polynomials $r^2 + \sigma r + 1$ and $r^2 + (\sigma - 1)r + 1$ have complex roots (otherwise exponential growth ensues). The only *integer* values of σ satisfying this criterion are $\sigma = -1, 0, 1, 2$. We remark that the three values mentioned above are exactly retrieved plus the value $\sigma = -1$. A close inspection of this mapping shows that it is also integrable: it is just a form of an ultra-discrete P_{III}, obtained from the discrete system $x_{n+1}x_{n-1} = x_n(x_n + \lambda^n)$ which leads to (9) with $\phi(n) = 0$.

We have applied the low-growth criterion to other cases like u-P_{II} and u-P_{III} and in every case the results of the growth analysis correspond to the already obtained integrable cases. However low-growth is not a sufficiently powerful integrability criterion. In particular the inhomogeneous terms (ϕ in equation (9)) cannot be fixed by this argument. Any slow-growing $\phi(n)$ would satisfy this requirement. So another criterion must complement this first one.

In the case of (continuous) evolution equations two integrability criteria are often used in conjunction: the Painlevé property and the existence of multisoliton solutions. In the case of Painlevé equations the latter are the special solutions that exist for particular values of the parameters [19] (except for P_I which is parameter-free). A particular class of these solutions (existing also in the discrete case) are the rational ones. We shall investigate this property in the case of u-P equations. This will strengthen the argument that (8) are indeed Painlevé equations and will allow us to fix completely their form. For d-P_{II} the simplest rational solution is a constant. Thus a constant solution should exist for u-P_{II}. Instead of working with (8d) we first transform it to a canonical form through a translation of X and a linear transformation in n:

$$X_{n+1} + X_{n-1} = a + (n - X_n)_+ - (n + X_n)_+ \tag{10}$$

We find indeed that $X = 0$ is a solution of (10) for $a = 0$. However this solution exists if we replace both n's in (10) by any function of n. The next solution for u-P_{II} is a step-function one. Indeed when n is large constant positive solution exists with X equal to $a/4$ while a constant solution with $X = a/2$ exists when n is large negative. Thus, for instance for $a = 4$ a solution for $n << 0$ is $X = 2$, while for $n >> 0$ a solution is $X = 1$. The remarkable fact is that that these constant "half" solutions do really join to form a solution of (10) with a unique jump at $n = -1$. It is straightforward to check that this will not be the case in general if the non-autonomous part is not linear in n. The general solution of this type becomes now clear. For $a = 4m$ we have a solution with m jumps from the value $X = 2m$ to $X = m$. The first jump occurs at $n_0 = 1 - 2|m|$ and we have successive jumps of $-|m|/m$ at $n = n_0 + 3k$, $k = 0, 1, 2, \ldots, |m| - 1$. Analogous results can be obtained for the other u-P_{II} (8e). Thus u-P_{II} has a rich structure of particular solutions.

Let us conclude by coming back to the question of whether the ultra-discrete equations we have derived are justifiably called Painlevé equations. When Painlevé equations were first introduced the aim was to extend the notion of function. While linear differential equations define the well-known special functions, the aim of Painlevé and his collaborators was to obtain new transcendents as solutions of nonlinear differential equations. Already for discrete Painlevé equations the question whether their solutions define new functions (in the appropriate space) has not been fully answered. What we are certain about is that the discrete Painlevé's (just as the continuous ones) satisfy the three major integrability criteria. These are: the existence of Lax pairs, the "Painlevé property" and the existence of "multisoliton" solutions. The first corresponds to a purely constructive approach while the second is based on the singularity structure and uses singlevaluedness as a necessary condition for the definition of a function. The third criterion is a practical one. Integrability is conjectured whenever there exists a rich class of solutions which are globally described and are interrelated by various transformations. In the case of ultra-discrete systems the application of these criteria does not lead to a clear answer as in the discrete case. The Painlevé criterion based on singularities is clearly inapplicable here because of the absence of singularities. Some progress has been recently done in the direction of Lax pairs [20] and could presumably be extended to ultra-discrete Painleveé equations. Fortunately, the third, more practical, criterion can be easily extended to the ultra-discrete domain. Work is in progress on this point and our first results [21] show that the ultra-discrete systems we introduced here do possess rich families of solutions which, moreover, are related by the ultra-discrete equivalent of Bäcklund transformations. Thus the analogy between what we called ultra-discrete Painlevé equations and their discrete and continuous homonyms is strengthened. A last argument is based on the systematic character of the ultra-discretisation. The procedure is not an *ad hoc* one but starts with an integrable discrete system and derives in an unambiguous way its ultra-discrte equivalent. Although tha above arguments do not constitute a proof that the cellular automaton Painlevé equations define indeed nonlinear transcendents in the ultra-discrete domain they do constitute some convincing evidence. We plan to further strengthen these arguments by studying the higher u-P's together with their properties.

The questions left open here (or at leats some of them) will be addressed in future works. What is important at this stage is that we have shown that this new domain of integrable systems is particularly rich. While the discrete systems are the fundamental entities and contain all the structure, the cellular automata are their bare-bones versions capturing the essence of the dynamics. This explains the interest that these ultra-discrete systems present both from the mathematical and the physical points of view.

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Figure captions.

Figure 1. Distance between the two particles as a function of time in the case (a) of the continuous Toda potential and (b) its ultra-discrete analog.