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Cellular Automaton to Optical Communication: Diversity of Solitons

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§1. Introduction

A quarter century has passed since Zabusky and Kruskal discovered solitons in the numerical computation of the Korteweg-deVries equation[1]. The concept of solitons has been widely introduced in various nonlinear wave systems. We now realize that solitons play important roles to transport energy, information and so on in the wave systems.

In this paper, we present some results on two extreme cases in which solitons arise: The cellular automaton which may be considered to be one of the simplest soliton system at present, and the optical communication system where the solitons are considered to be a quite advantageous entity.

§2. Soliton Cellular Automaton

Cellular automata (CA's) are simple models which are introduced to understand the complex physical, chemical and biological phenomena[2]. In some cases of the one-dimensional CA, strings of bits behave like solitons, namely interact with one another preserving their identities. The filter automaton which has been proposed by Park, Steiglitz and Thurston is one of the typical system where soliton collisions are quite common[3-5].

Recently, Takahashi and Satsuma proposed a CA of the filter automaton type in which any state *only* consists of solitons[6]. The CA is one-time and one-space dimensional and two-valued. The evolution rule for the CA is given as follows: Let u_j^t be 1 or 0, which denotes the value at integer time t and at integer space site j. Ranges of t and j are both from $-\infty$ to $+\infty$ and it is assumed that $u_j^t = 0$ for j far enough to the left and to the right. The variable u_j^t obeys

$$u_j^{t+1} = \begin{cases} 1 & \text{if } u_j^t = 0 \text{ and } \sum_{i=-\infty}^{j-1} u_i^t > \sum_{i=-\infty}^{j-1} u_i^{t+1} \\ 0 & \text{otherwise} \end{cases} . \tag{1}$$

It is to be noted that this rule is spatially nonlocal. If we introduce S_j^t by $S_j^t = \sum_{i=-\infty}^{j} u_i^t$, then the rule (1) may be rewritten by

$$S_j^{t+1} - S_{j-1}^{t+1} = \frac{1}{2\delta}(1 - S_j^t + S_{j-1}^t)(S_{j-1}^t - S_{j-1}^{t+1} + \delta - |S_{j-1}^t - S_{j-1}^{t+1} - \delta|) , \tag{2}$$

where δ is a constant satisfying $0 < \delta \leq 1$.

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$j \rightarrow$

$t = 0$	$\cdots$00001111000000000001100011000001000000000000000000000000000000$\cdots$
1	$\cdots$00000000111100000000011000110000100000000000000000000000000000$\cdots$
2	$\cdots$00000000000011110000000110001100010000000000000000000000000000$\cdots$
3	$\cdots$00000000000000001111000001100011001000000000000000000000000000$\cdots$
4	$\cdots$00000000000000000000111100011000110100000000000000000000000000$\cdots$
5	$\cdots$00000000000000000000000011100111001011000000000000000000000000$\cdots$
6	$\cdots$00000000000000000000000000011000110100111100000000000000000000$\cdots$
7	$\cdots$00000000000000000000000000000110001011000011110000000000000000$\cdots$
8	$\cdots$00000000000000000000000000000001100100110000001111000000000000$\cdots$
9	$\cdots$00000000000000000000000000000000011010001100000000111100000000$\cdots$
10	$\cdots$00000000000000000000000000000000000101100011000000000011110000$\cdots$

Figure 1. Example of time evolution according to eq.(1)

There exist two types of solitary waves in this CA. The first is the "simple solitary wave" which consists of only one sequence of an arbitrary finite number of 1's. The speed of this wave is equal to the number of 1's. The other is the "compound solitary wave" which includes an arbitrary finite number of simple solitary waves of the same length and apart from one another at a distance equal to or greater than the length. The speed of this wave is equal to the length of the simple solitary waves included.

Figure 1 shows an example of interaction among two simple solitary waves and one compound solitary wave. We see that three waves recover their identities after the collision. The only effect of collision is the phase shift. We define a phase shift of each wave by the shift of its site number caused by the interactions with other waves. The faster solitary wave gains the phase shift of the double of the number of 1's of the slower one, and the slower loses the same amount. Hence these solitary waves are considered to be solitons. In fact, we can prove that all solitary waves conserve their identity in the time evolution (see [6] for the detail).

One of the remarkable properties of soliton systems is the existence of an infinite number of conserved quantities. The CA we are discussing also has such quantities. Let us difine the number of the basic strings as follows; $\sharp 101$ is the number of 101 in the sequence at a certain time. For example, $\sharp 101$ is 2 for the sequence, $\cdots 00011010100100 \cdots$. It is not so hard to prove for the CA that $\sharp 1$, $\sharp 11$, $\sharp 1101 + \sharp 0010$, $\sharp 110101 + \sharp 001010$, and so on conserve in the time evolution.

As we mentioned before, any state of the CA consists only of the solitons. Moreover, we can identify solitons for any state of the CA, which means that the initial value problem is solved exactly. Therefore, this CA may be considered to be one of the simplest analog of soliton equation. We remark that the spatial nonlocality of the rule of time evolution may yield such a remarkable property.

Finally in this section, we give another representation of this CA. Let us introduce the multi-valued variable v_j^t by

$$v_j^t = v_{j-1}^t + \begin{cases} +1 & \text{if } u_j^t = 1 \\ -1 & \text{if } u_j^t = 0 \text{ and } v_{j-1}^t \neq 0 \\ 0 & \text{otherwise} \end{cases} . \tag{3}$$

Then, for example, we have the following correspondence between u_j^t and v_j^t;

u_j^t $\cdots 000000000111001011000000000 \cdots$

v_j^t $\cdots 000000000123212123210000000 \cdots$

§3. Optical Soliton with Two Peaks

One of the most succesful applications of soliton phenomena in engineering is optical communication. Since Hasegawa and Tappert discussed the possibility of soliton propagation in optical fibers and the prediction was cofirmed by experiment later on, many theoretical and experimental studies have been done to achieve a communication system based on optical solitons[7]. Among them, the higher-order nonlinear Schrödinger (HNLS) eqation,

$$i\frac{\partial q}{\partial T}+\frac{1}{2}\frac{\partial^2 q}{\partial X^2}+|q|^2q+i\epsilon\{\beta_1\frac{\partial^3 q}{\partial X^3}+\beta_2|q|^2\frac{\partial q}{\partial X}+\beta_3 q\frac{\partial |q|^2}{\partial X}\}=0, \tag{4}$$

is proposed to understand the higher order effect which can not be explained by the NLS equation ($\epsilon=0$ in eq.(4))[8,9]. There are a few cases where eq.(4) is exactly solvable; (i) the derivative NLS equation-typeI ($\beta_1 : \beta_2 : \beta_3 = 0:1:1$), ($ii$) the derivative NLS equation-typeII ($\beta_1 : \beta_2 : \beta_3 = 0:1:0$), and ($iii$) the Hirota equation ($\beta_1 : \beta_2 : \beta_3 = 1:6:0$).

Very recently, Sasa and Satsuma have shown that eq.(4) is also solvable by the inverse scattering transform if the condition, $\beta_1 : \beta_2 : \beta_3 = 1:6:3$, is imposed[10]. In this section, we briefly comment on the result and discuss about the property of the soliton solution.

In order to analyze eq.(4) with the coefficients, $\beta_1=1$, $\beta_2=6$ and $\beta_3=3$, it is rather convenient to introduce variable transformations,

$$u(x,t)=q(X,T)\exp\{\frac{-i}{6\epsilon}(X-\frac{T}{18\epsilon})\},\quad t=T,\quad x=X-\frac{T}{12\epsilon}. \tag{5}$$

Then, eq.(4) is reduced to a complex modified KdV-type equation,

$$\frac{\partial u}{\partial t}+\epsilon(\frac{\partial^3 u}{\partial x^3}+6|u|^2\frac{\partial u}{\partial x}+3u\frac{\partial |u|^2}{\partial x})=0, \tag{6}$$

on which we can formulate the inverse scattering scheme written by the 3×3 Lax pair.

The one-soliton solution obtained by using the inverse scattering transform may be written in the following form:

$$q(X,T)=\frac{\eta e^{iB}\{2\cosh A+(c-1)e^{-A}\}}{\cosh(2A-\log|c|)+|c|}, \tag{7}$$

where

$$A=\eta[X-\{\xi-\epsilon(\eta^2-3\xi^2)\}T-X^{(0)}], \tag{8a}$$

$$B=\xi[X+\{(\eta^2-\xi^2)/(2\xi)+\epsilon(\xi^2-3\eta^2)\}T-X^{(1)}], \tag{8b}$$

$$c=1-i\eta/\{\xi-1/(6\epsilon)\}. \tag{8c}$$

If we take the limit of $\epsilon\to 0$ ($c\to 1$), eq.(7) reduces to the one-soliton solution of the NLS equation as is expected. On the other hand, if we take the limit of $\xi\to 1/(6\epsilon)$ ($|c|$

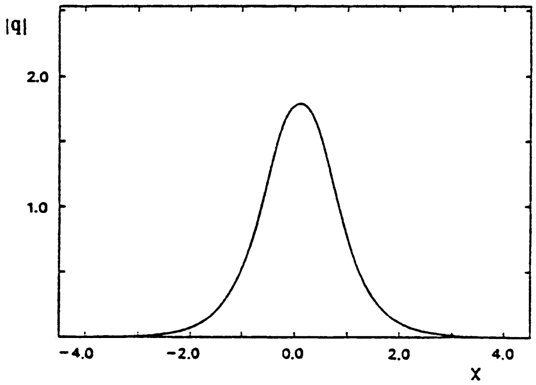

Figure 2a. $|q|$ for $\eta = 2$ and $|c| = 1.5$.

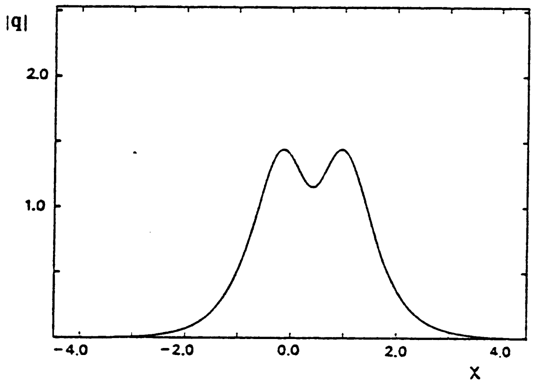

Figure 2b. $|q|$ for $\eta = 2$ and $|c| = 5.0$.

$\to \infty$), eq.(7) reduces to another sech-type solution,

$$q(X,T) = \frac{\eta}{\sqrt{2}} \exp\{\frac{i}{6\epsilon}(X - \frac{1}{18\epsilon}T - X^{(1)})\} \text{sech}[\eta\{X - (\epsilon\eta^2 + \frac{1}{12\epsilon})T - X^{(0)}\} + \log\sqrt{2}]. \quad (9)$$

We note that eq.(9) does not tend to the NLS solution in the limit fo $\epsilon \to 0$. In a sense, eq.(9) gives a singular soliton solution for the perturbed NLS equation.

The shape of $|q|$ depends on the parameters η and $|c|$. If $1 \leq |c| \leq 2$, $|q|$ has only one maximum (Fig.2a). Note that $|c| = 1$ corresponds to $\epsilon = 0$ which is the case of the NLS equation. On the other hand, if $|c| > 2$, the soliton solution shows an interesting feature. In this case, $|q|$ has two maxima (Fig.2b). This type of soliton has not yet been reported for the higher-order NLS equation. We remark that, if we take the limit of $|c| \to \infty$, the hump at the right-hand side moves to infinity, and the hump remained becomes the soliton solution (9).

We have shown by means of the inverse scattering transform that the higher-order NLS eq.(4) has an interesting soliton solution which propagates steadily with two peaks of the same height. Interesting problems are in what condition such a soliton appears and whether the soliton is stable or not under certain perturbations. Finally in this section, we show the partial result on the problem by using the numerical computation.

We have solved eq.(4) with $\beta_1 = 1$, $\beta_2 = 6$ and $\beta_3 = 3$ numerically for the initial value $q(X, T = 0) = \text{sech}X$. If $\epsilon = 0$, this initial value just gives the one-soliton solution of the NLS equation, and therefore $|q|$ does not change in the time evolution. Figure 3 shows the numerical result for $\epsilon = 0.1$. We see only a slight change of the initial wave form. The behavior is comparatively similar to that of the NLS solitons.

On the other hand, we have a quite different behavior if ϵ is large. Figure 4 shows the numerical result for $\epsilon = 0.5$. We see that the sech-type solitons appear in the course of time again, although the shape is different from that in Fig.3. We interpret this result as follows: In this case, the second and third terms in eq. (4) may be ignored. Then, since eq. (4) is regarded as the modified KdV equation approximately, the solitons in Fig.4 are close to those of the equation.

The numerical result for the intermediate value of ϵ ($\epsilon = 0.195$) is given in Fig.5. The soliton with two peaks seems to appear. However, when we eliminated the ripple part at $T = 30$ and calculated further, we observed that it behaves like the breather solution. This means that the observed pulse is not the soliton (7) itself. However, our result shows the chacteristic structure of the soliton with two peaks is seen in the numerical solution for this value of ϵ.

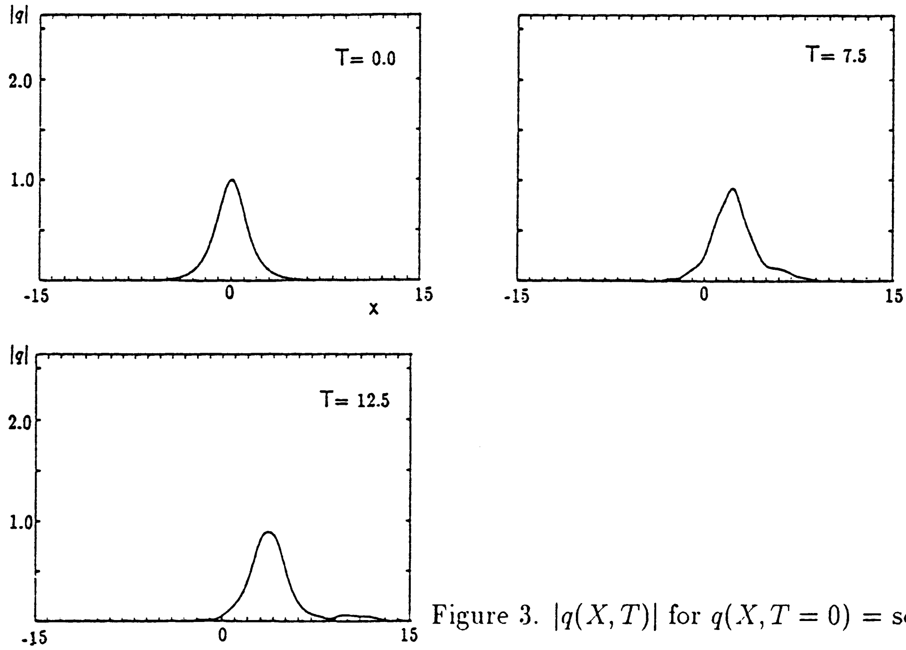

Figure 3. $|q(X,T)|$ for $q(X,T=0) = \text{sech}X$ ($\epsilon = 0.1$).

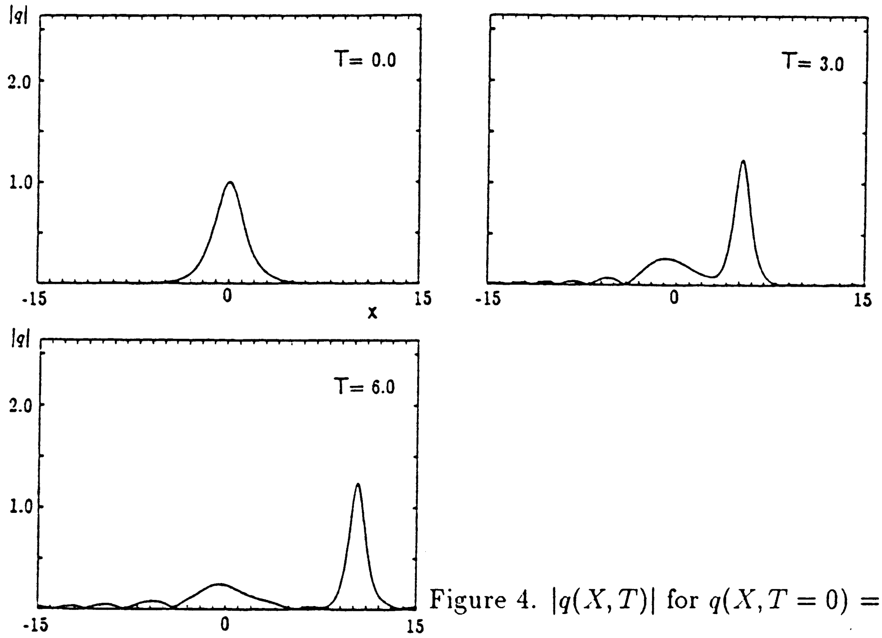

Figure 4. $|q(X,T)|$ for $q(X,T=0) = \text{sech}X$ ($\epsilon = 0.5$).

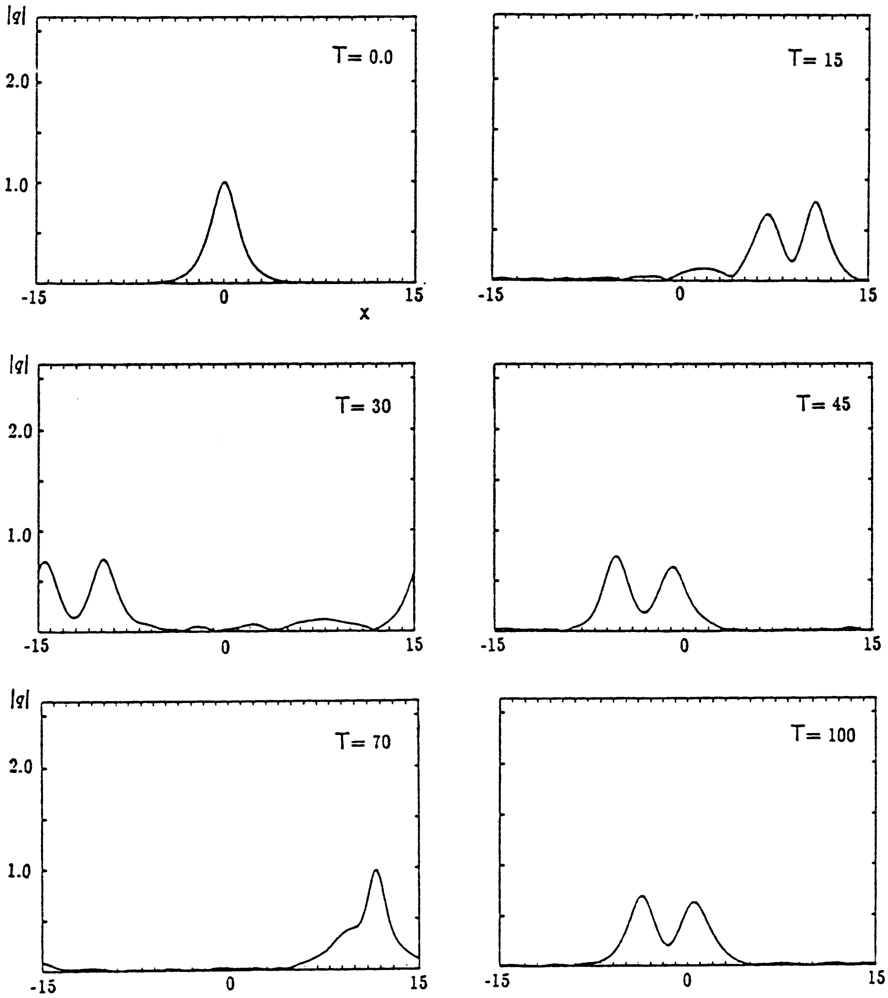

Figure 5. $|q(X,T)|$ for $q(X,T=0) = \text{sech}X$ ($\epsilon = 0.195$). The ripple part is eliminated at $T = 30$.

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