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Three-dimensional fundamental diagram of particle system of 5 neighbors with two conserved densities

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Abstract

We discuss a particle system of 5 neighbors with two independent conserved densities. The mean momentum uniquely depends on a pair of the densities and a three-dimensional fundamental diagram is obtained. It shows the phase transition of behavior of asymptotic solution to the system. Moreover, we propose two other systems which have the similar unique dependency obtained numerically.

Keywords fundamental diagram, particle system, conserved density

Research Activity Group Applied Integrable Systems

1. Introduction

Fundamental diagram is an important object showing the asymptotic behavior of solutions to the system where particles move in the space sites [1,2]. It shows the relation between the density of particles and their mean momentum. There are interesting phenomena 'phase transition' such that the momentum suddenly changes at some critical points of density. For example, a phase transition from free flow to congested flow in a traffic system is well-known and has been studied theoretically using various mathematical models [3–5].

However, the density is not a unique parameter to determine the phase of particle flow. There are many systems which give different mean momenta according to the variation of initial data for the same density [6]. To solve this redundancy, some other quantities should be necessary to determine the momentum in addition to the density.

Our purpose of this letter is to show a three-dimensional framework to the fundamental diagram using two conserved densities [7]. First, we propose an exact analysis on solutions to a particle system and derive a three-dimensional fundamental diagram. The mean momentum is uniquely determined by a pair of particle density and another conserved density. Second, we report two other relevant systems of which similar diagrams are obtained numerically.

2. Particle system with two conserved quantities

Let us consider the time evolution equation of a particle system as follows:

$$u_j^{n+1} = u_j^n + q(u_{j-2}^n, u_{j-1}^n, u_j^n, u_{j+1}^n) \\ - q(u_{j-1}^n, u_j^n, u_{j+1}^n, u_{j+2}^n), \quad (1)$$

Table 1. Rule table of $q(a,b,c,d)$. Upper and lower rows denote (a,b,c,d) and $q(a,b,c,d)$ respectively.

1111	1110	1101	1100	1011	1010	1001	1000
1	1	1	1	0	0	0	0

0111	0110	0101	0100	0011	0010	0001	0000
0	1	0	0	0	0	0	0

where j is a site number, n is an integer time, $u \in \{0, 1\}$ is a binary state value and the flux $q(a, b, c, d)$ is given by the binary table shown in Table 1. If $a = b = 1$ or $(a, b, c, d) = (0, 1, 1, 0)$, $q(a, b, c, d) = 1$ and otherwise 0. Assume a periodic boundary condition for space sites with period L, that is, $u_{j+L}^n = u_j^n$.

Consider that $u_j^n = 1$ denotes a particle existing at site j and time n and $u_j^n = 0$ an empty site. Since (1) follows the conservation form, $\sum_{j=1}^L u_j^n$ is constant for n. Thus, the density $\rho = \sum_{j=1}^L u_j^n / L$ is also. Moreover, flux $q(u_{j-2}^n, u_{j-1}^n, u_j^n, u_{j+1}^n)$ means the number of particles moving from site $j - 1$ to j from n to $n + 1$.

Eq. (1) means the following motion rule of particles.

- An isolated particle (010) does not move.

- For a pair of adjacent two particles (0110), both particles move.

- For a sequence of more than two particles (011...10), particles other than the leftmost move.

Fig. 1 shows the motion of particles and the state of adjacent empty sites from time n to $n + 1$. Note that the symbol $*$ denotes an indefinite site of which value is dependent on values of other sites at time n. Fig. 2 shows two examples of solution to (1).
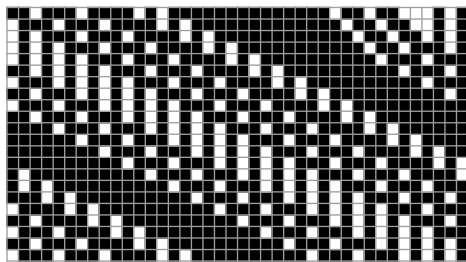
There exists another conserved quantity for (1). Let us define $\#(x_1 x_2 \ldots x_k)^n$ by the number of local patterns $x_1 x_2 \ldots x_k$ included in the space sites at time n.

n :   010   0110   01110   011110   ...
$n+1$:   *10   *011   *1011   *10111   ...

Fig. 1. Motion of particles.
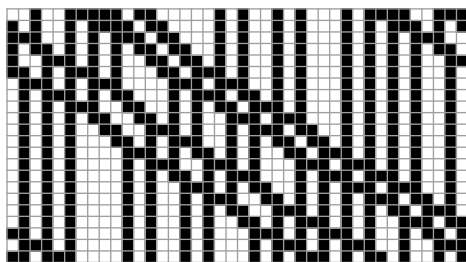
(a) $\rho = 0.5$

(b) $\rho = 0.75$

Fig. 2. Examples of solutions to (1). Black square ■ and white square □ denote $u = 1$ and 0 respectively. Initial data is shown at the top line and the state evolves downwards.

n :   0111*   0110
$n+1$:   *1011   *011

Fig. 3. Two patterns which produce 011 at the next time.

Then, $\#011^n$ is a conserved quantity since the pattern 011 at time $n+1$ are produced only from the patterns 0111 or 0110 as shown in Fig. 3. Thus we obtain $\#011^{n+1} = \#0111^n + \#0110^n = \#011^n$. Note that $\#1^n = \sum_j u_j^n$ is conserved and $\#0^n = L - \#1^n$ also. Moreover, $\#110^n = \#011^n$ since $\#110^n + \#010^n = \#10^n$, $\#011^n + \#010^n = \#01^n$ and $\#01^n + \#11^n = \#10^n + \#11^n = \#1^n$ always hold. Therefore, there are two independent conserved quantities $\#1^n$ (or $\#0^n$) and $\#011^n$ (or $\#110^n$) for (1). We omit their superscript n as $\#1$ ($\#0$) and $\#011$ ($\#110$) to indicate their conservation.

3. Asymptotic behavior

Let us introduce a notation $0k0$ ($k \in \mathbb{N}$) which means a local pattern of a sequence of 1's of length k neighbored on both sides by 0. For example, $020 = 0110$ and $\#030^n = \#01110^n$. We obtain the following lemma on $\#0k0^n$ for $k \geq 4$.

Lemma 1 *For large enough n ($n \gg 0$), $\#0k0^n$ ($k \geq 4$) of any solution to (1) becomes constant for n.*

The proof of this lemma is as follows. Every 1 moves to the right by one site or stays at the same position for each time. Every stationary 1 is neighbored to the right by 0 at the next time from Fig. 1. Moreover, 1's other than the leftmost move to the right for the sequence $0k0$ ($k \geq 4$). Thus $\#0k0^n$ decreases monotonically for

n :   0111...11010   0111...110111*
$n+1$:   *1011...1110   *1011...111011

Fig. 4. Two cases that $0k0$ keeps its length.

n :   0111...110010   0111...1100111*
$n+1$:   *1011...11010   *1011...1101011
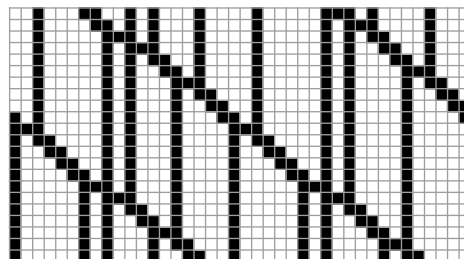
Fig. 5. Decrease of length of 111...11 adjacent to 00.

Fig. 6. Merging and splitting among local patterns 0110, 010 and 030.

n, converges for $n \gg 0$ and the lemma holds. Note that the lemma does not hold for $k \leq 3$ since both 1's moves to the right for 0110.

If there exists a positive K such that $K = \max\{k | k \geq 4, \#0k0^n > 0, n \gg 0\}$, the sequences $0K0$ can be maintained by the following two types of evolutions as shown in Fig. 4. In both cases, if a sequence of two or more 0's exists, $\#0k0^n$ decreases as shown in Fig. 5, thus such a sequece can not be included in the asymptotic solution due to Lemma 1. Similar argument is applied to the sequences for $0k0$'s ($4 \leq k < K$) and we can easily show the following lemma.

Lemma 2 *Assume that one or more local patterns 040, 050, ... exist for $n \gg 0$. Then, they move to the right by two sites for every time. All other 1's are included in 010 or 030. All 0's are isolated, that is, exist as 101. Any $\#0k0^n$ ($k \in \mathbb{N}$) is constant for n in this asymptotic solution.*

There is a special case of asymptotic solutions added to those described in Lemma 2. If all 1's are included in the local patterns 030 or 010 and all 0's are isolated, the state is an asymptotic solution and any $\#0k0^n$ is constant.

To summarize asymptotic solutions of this special case and of Lemma 2, they can be realized by the condition $\#0110^n = \#00^n = 0$. Therefore, $\#0110^n + \#00^n > 0$ is necessary for another type of asymptotic solution. Moreover, $\#0k0^n = 0$ holds for any $k \geq 4$ from Lemma 2 in the latter type of solution. Since 0110 and 010 can merge and produce 030 and conversely 030 can split into 0110 and 010, description of asymptotic behavior of this type of solution is more complicated than that of the former type. Fig. 6 shows an example of solution where merging and splitting among 0110, 010 and 030 occur.

However, the conditions about two types of asymptotic solutions shown above provide enough information to derive a three-dimensional fundamental diagram. We give the following theorem to distinguish two types of asymptotic solutions as a consequence of this section.

Theorem 3 *Asymptotic solution to* (1) *for $n \gg 0$ obeys either of the following conditions.*

(A) $\#0110^n = \#00^n = 0$.

(B) $\#0110^n + \#00^n > 0$ *and* $\#0k0^n = 0$ *for any $k \geq 4$.*

4. Three-dimensional fundamental diagram

First, we show that two types (A) and (B) described in Theorem 3 are distinguished by the signature of $2\#011 - \#1 + \#0$. The following formulas always hold for any state.

$$\#1 = \sum_{k \geq 1} k \#0k0^n, \quad \#0 = \sum_{k \geq 1} \#0k0^n + \#00^n.$$

For type (A), using these formulas and noting $\#0110^n = \#00^n = 0$ we have

$$2\#011 - \#1 + \#0$$
$$= 2 \sum_{k \geq 3} \#0k0^n - \left(\#010^n + \sum_{k \geq 3} k \#0k0^n \right)$$
$$+ \left(\#010^n + \sum_{k \geq 3} \#0k0^n \right)$$
$$= - \sum_{k \geq 4} (k-3) \#0k0^n \leq 0.$$

For type (B), noting $\#0110^n + \#00^n > 0$ and $\#0k0^n = 0$ for any $k \geq 4$, we have

$$2\#011 - \#1 + \#0$$
$$= 2(\#0110^n + \#030^n)$$
$$- (\#010^n + 2\#0110^n + 3\#030^n)$$
$$+ (\#010^n + \#0110^n + \#030^n + \#00^n)$$
$$= \#0110^n + \#00^n > 0$$

Therefore, the type (A) is in the case of $2\#011 \leq \#1 - \#0$ and (B) $2\#011 > \#1 - \#0$.

The mean momentum Q is given by counting the number of moving particles as follows.

$$Q = \frac{1}{L} \left(\sum_{k \geq 3} (k-1) \#0k0^n + 2\#0110^n \right).$$

We have

$$Q = \frac{1}{L} \left(\sum_{k \geq 3} (k-1) \#0k0^n \right) = \frac{1}{L} (\#1 - \#0)$$

for type (A) and

$$Q = \frac{1}{L} (2(\#030^n + \#0110^n)) = \frac{2}{L} \#011 = 2\rho_{011}$$

for type (B) considering the conditions for both types.

Finally, the following relation about the fundamental diagram is derived.

Theorem 4 *The mean momentum of* (1) *is uniquely given by two independent conserved densities $\rho = \#1/L$ and $\rho_{011} = \#011/L$ as follows.*

$$Q = \max(\underbrace{2\rho - 1}_{\text{type (A)}}, \ \underbrace{2\rho_{011}}_{\text{type (B)}}).$$
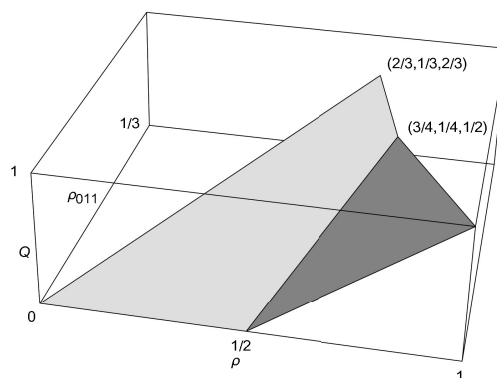

Fig. 7.   Fundamental diagram of (1).

Fig. 7 shows the three-dimensional fundamental diagram obtained by Theorem 4. There is a region of ρ where both asymptotic solutions of type (A) and (B) appear from different initial data. Therefore, another conserved density ρ_{011} is a key density to distinguish them.

5. Other systems

There are many other particle systems in the form of (1) with q different from that of Table 1. We made numerical experiments on such systems and found some may have a three-dimensional fundamental diagram dependent on a conserved density or a quasi-conserved density in addition to the density ρ. We show a rule table of $q(a,b,c,d)$ and a predicted Q of two examples below.

Flux $q(a,b,c,d)$ of the first example is given by the following rule table.

1111	1110	1101	1100	1011	1010	1001	1000
0	1	1	1	0	0	0	0

0111	0110	0101	0100	0011	0010	0001	0000
0	1	0	0	0	0	0	0

From numerical results, we predict $Q = \min(2(1-\rho), \rho - \rho_{\text{odd}})$ where ρ_{odd} is defined by $(\sum_{k=1}^{\infty} \#0(2k-1)0)/L$, that is, the number of sequences of 1's of odd length. The density ρ_{odd} is conserved in this system since we can easily show any movable 1 is either of 1's included in a local pattern 110.

Flux $q(a,b,c,d)$ of the second example is given by the following rule table.

1111	1110	1101	1100	1011	1010	1001	1000
0	1	1	1	0	1	0	1

0111	0110	0101	0100	0011	0010	0001	0000
0	0	1	1	-1	0	0	0

We predict $Q = \min(2\rho, -4\rho_{1*0} + 1, 2(1 - \rho))$ where ρ_{1*0} is the density of the number of local patterns 110 or 100. Though the density ρ_{1*0} is not a conserved density exactly, it becomes constant in the asymptotic solution for $n \gg 0$ from any initial data considering the evolution rule.

6.   Concluding remarks

We made an exact analysis on a particle system of 5 neighbors in the conservation form using a flux. The mean momentum is uniquely determined by two independent conserved densities and it provides a three-dimensional fundamental diagram. Moreover, we proposed similar other systems using numerical experiments.

One of future problems is to analyze them exactly and propose a common mechanism to give the three-dimensionality of fundamental diagram. Moreover, there exist particle systems which have three or more conserved densities. It is another future problem to find a higher dimensional dependency of mean momentum on conserved densities.

Theoretical analysis on asymptotic behavior of particle systems can be applied to real transportation systems. It is expected that information on mean momentum detailed by multiple conserved densities provides useful solution for transportation problems.

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