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# Max-plus equation with two conserved quantities and one monotonically decreasing quantity

Yuji Tokieda[1] and Daisuke Takahashi[1]

[1] Department of Pure and Applied Mathematics, Waseda University, 3-4-1 Okubo, Shinjuku-ku, Tokyo 169-8555, Japan

E-mail *m1r-i5t-ka1t@fuji.waseda.jp, daisuket@waseda.jp*

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**Abstract**

We propose a max-plus equation as an extension of 1+1D cellular automaton of four neighbors. It has two conserved quantities and one monotonically decreasing quantity which are max-plus analogues of those of the binary cellular automaton. It is interesting that one of the quantities is global and defined by the maximum value of dependent variable. We show proofs about their conservation and monotonicity using max-plus formulas.

**Keywords** max-plus equation, particle system, cellular automaton, conserved quantity

**Research Activity Group** Applied Integrable Systems

## 1. Introduction

Discrete equations have been used as mathematical models for diverse natural or social phenomena. There are various levels of discreteness, for example, difference equation, coupled map lattice, cellular automaton (CA), and so on. Among them, max-plus equation is quite novel expression and studied intensively in recent years [1–6].

Max-plus equation is a piecewise linear type of difference equation constructed from 'max', 'min', '+' and '−' operations on real number. Since its dependent variable can be closed in the range of integer due to piecewise linearity, max-plus equation is often called 'ultradiscrete equation'. Moreover, it is derived by the limiting procedure called 'ultradiscretization' from the usual type of difference equation defined by arithmetic operations [1]. For example, if we apply a transformation of dependent variable to the discrete Burgers equation using an exponential function with a small parameter and take a limit of the parameter, we obtain a max-plus equation called 'ultradiscrete Burgers equation' [2]. The range of its dependent variable can be confined to binary value $\{0,1\}$. Thus, the ultradiscrete Burgers equation reduces to an elementary CA of rule number 184 [7].

As for this example, CA is naturally embedded in the ultradiscrete Burgers equation with a special range of dependent variable, and the ultradiscrete Burgers equation is derived by ultradiscretizing the discrete Burgers equation approximating the (differential) Burgers equation. Therefore, we can relate various discrete level of equations with one another from fully continuous to fully discrete, even digital. In this context, max-plus equations propose a novel view to digital systems like CA.

In this letter, we focus on a certain $1+1$D binary CA of four neighbors with a primary conserved quantity. Assume that $j$ denotes integer space site number, $n$ integer time and $u_j^n$ the state value at site $j$ and time $n$. The value of binary dependent variable $u$ is 0 or 1. The general form of evolution rule is written by

$$u_j^{n+1} = f(u_{j-2}^n, u_{j-1}^n, u_j^n, u_{j+1}^n).$$

Define $\#(x_1x_2\dots x_k)^n$ ($x_i \in \{0,1\}$) by the total number of local pattern $x_1x_2\dots x_k$ in the space sites at time $n$. Note that if $\#(x_1x_2\dots x_k)^n$ is conserved, that is, constant for $n$, we omit the superscript $n$ as $\#(x_1x_2\dots x_k)$. There are special type of rules such that $\#1$ is conserved. We call such a CA 'particle CA' (PCA) after the reference [8]. In the reference, PCA's of four neighbors are called 'PCA4' and there exist four independent rules from PCA4-1 to 4-4. Moreover, it is reported that PCA4-1, 4-2 and 4-3 are embedded in the max-plus equations respectively of which initial value problem can be solved. However, an appropriate max-plus equation for PCA4-4 of which general solution can be derived has not yet been found.

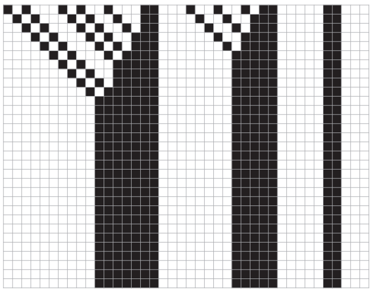
In this letter, we propose a max-plus equation which includes PCA4-4 as a binary case. PCA4-4 has another conserved quantity of higher order and a monotonically decreasing quantity other than the primary conserved quantity $\#1$. The max-plus equation proposed has counterparts for all those quantities.

Contents of this letter are as follows. In Section 2, we explain the definition of PCA4-4 and its quantities described above. In Section 3, we propose a max-plus equation obtained by extending PCA4-4 and show it has counterparts of quantities. In Section 4, we give concluding remarks.

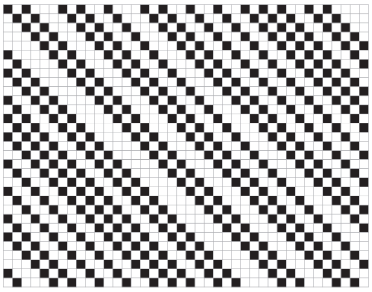
## 2. Definition and properties of PCA4-4

PCA4-4 can be written in the following conservation form [9],

$$u_j^{n+1} = u_j^n + q(u_{j-2}^n, u_{j-1}^n, u_j^n) - q(u_{j-1}^n, u_j^n, u_{j+1}^n), \quad (1)$$



$\rho = 0.35$



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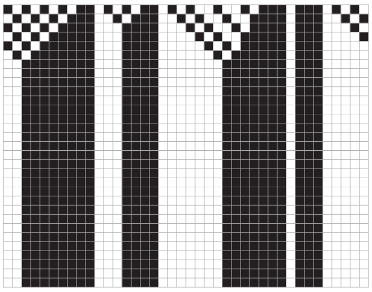

$\rho = 0.55$

Fig. 1. Examples of solution to PCA4-4 ($u_j^n \in \{0, 1\}$).

where $q(a, b, c) = 1$ if $(a, b, c) = (0, 1, 0)$ and 0 otherwise. Below, we consider initial value problem about (1) with initial time $n = 0$ and assume a periodic boundary condition for a finite number of space sites. Under this boundary condition, PCA4-4 has two types of steady solutions at large enough $n$ depending on the initial data, one is a right-moving pattern $u_j^{n+1} = u_{j-1}^n$ and the other is a stationary pattern $u_j^{n+1} = u_j^n$. If the initial data contains at least one local pattern '11', then the solution becomes stationary at large enough $n$. If there are no consecutive 1's in the initial data, the steady solution becomes right-moving. Fig. 1 shows examples of solution to PCA4-4 for different initial data.

Moreover, PCA4-4 has the following two conserved quantities and one monotonically decreasing quantity.

**Properties of PCA4-4**

(I) #1 is conserved.
(II) $\#010^n$ decreases monotonically.
(III) #110 (#011) is conserved.

We can easily show (I) since (1) is in the conservation form. Then we can consider 1's in the solution as moving particles in the space sites and $q(u_{j-1}^n, u_j^n, u_{j+1}^n)$ as a flux between the sites $j$ and $j+1$ at time $n$. Since the particle can move only if it is isolated ($q(0, 1, 0) = 1$), we can show properties (II) and (III) from this particle motion.

## 3. Max-plus equation extended from PCA4-4

In this section, we propose an extension of PCA4-4 using max-plus expression. We call this extended system 'Max4-4'. Note that Max4-4 includes PCA4-4 as a special case if the dependent variable $u$ is restricted to $\{0, 1\}$. The extension is realized by describing the flux $q(a, b, c)$ of (1) by max-plus expression. However, there exist a large freedom of expression. Thus, we assume that Max4-4 satisfies some corresponding properties (I), (II) and (III) of PCA4-4.

We propose the following max-plus equation as Max4-4.

$$\begin{aligned} &u_j^{n+1} = u_j^n + q(u_{j-2}^n, u_{j-1}^n, u_j^n) - q(u_{j-1}^n, u_j^n, u_{j+1}^n), \\ &q(a, b, c) = \max(0, \min(b - a, b - c)). \end{aligned} \tag{2}$$

If we restrict the dependent variable $u$ is 0 or 1, then $q(a, b, c) = 1$ if $(a, b, c) = (0, 1, 0)$ and 0 otherwise. Therefore (2) includes PCA4-4 as a special case. Fig. 2 shows examples of solution to Max4-4 with real initial data.

Before discussing the counterparts of properties of PCA4-4, we show that the range of value of solutions to (2) is closed and determined by the initial data. If $L$ and $M$ denote $\max_j\{u_j^0\}$ and $\min_j\{u_j^0\}$, respectively, we can derive $M \le u_j^n \le L$ holds for any $j$ and $n$ as follows. From (2), we have

$$\begin{aligned} u_j^{n+1} &= u_j^n + \max(0, \min(u_{j-1}^n - u_{j-2}^n, u_{j-1}^n - u_j^n)) \\ &\quad - \max(0, \min(u_j^n - u_{j-1}^n, u_j^n - u_{j+1}^n)) \\ &= \max(u_j^n, \min(u_j^n + u_{j-1}^n - u_{j-2}^n, u_{j-1}^n)) \\ &\quad - \max(0, \min(u_j^n - u_{j-1}^n, u_j^n - u_{j+1}^n)) \\ &\le \max(u_j^n, \min(u_j^n + u_{j-1}^n - u_{j-2}^n, u_{j-1}^n)) \\ &\le \max(u_{j-1}^n, u_j^n). \end{aligned}$$

We also obtain

$$\begin{aligned} u_j^{n+1} &= \max(0, \min(u_{j-1}^n - u_{j-2}^n, u_{j-1}^n - u_j^n)) \\ &\quad + \min(u_j^n, \max(u_{j-1}^n, u_{j+1}^n)) \\ &\ge \min(u_j^n, \max(u_{j-1}^n, u_{j+1}^n)) \\ &= \max(\min(u_{j-1}^n, u_j^n), \min(u_j^n, u_{j+1}^n)) \\ &\ge \min(u_{j-1}^n, u_j^n). \end{aligned}$$

Thus, we have

$$\max_j\{u_j^{n+1}\} \le \max_j\{u_j^n\}, \quad \min_j\{u_j^{n+1}\} \ge \min_j\{u_j^n\},$$

and

$$M \le u_j^n \le L,$$

for any $j$ and $n$. Therefore, the solution does not diverge from any initial data with a finite range. For example, if we use integer values from $M$ to $L$ for initial data, the value range of solution is also the same and (2) becomes CA with multiple same values.

Next, we show the counterparts of properties (I), (II) and (III) for Max4-4.

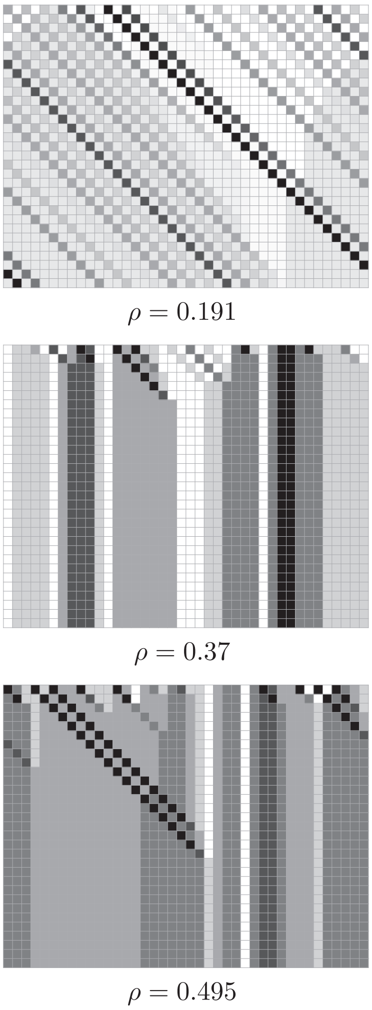


Fig. 2. Examples of solution to Max4-4 ($u_j^n \in \mathbb{R}$).

### *3.1 Property (I)*

The conserved quantity #1 of PCA4-4 can be counted by $S^n = \sum_j u_j^n$ and we consider this quantity is a counterpart of Max4-4 as it is. It is clear that $S^n$ is conserved for (2) since the evolution equation follows the conserved form under periodic boundary condition.

### *3.2 Property (II)*

About the property (II), $\#010^n$ is equal to the sum of flux for PCA4-4 since $q(a,b,c) = 1$ if and only if $(a,b,c) = (0,1,0)$. Therefore, we can consider its counterpart of Max4-4 is defined by

$$Q^n = \sum_j q(u_{j-1}^n, u_j^n, u_{j+1}^n).$$

To discuss the monotonicity of $Q^n$, let us consider

$$Q^n - Q^{n+1} = \sum_j q(u_{j-1}^n, u_j^n, u_{j+1}^n) - \sum_j q(u_{j-1}^{n+1}, u_j^{n+1}, u_{j+1}^{n+1}).$$

Using (2), the above equation can be rewritten in the following form,

$$Q^n - Q^{n+1} = \sum_j (\max A_j^n \cup B_j^n - \max A_j^n), \qquad (3)$$

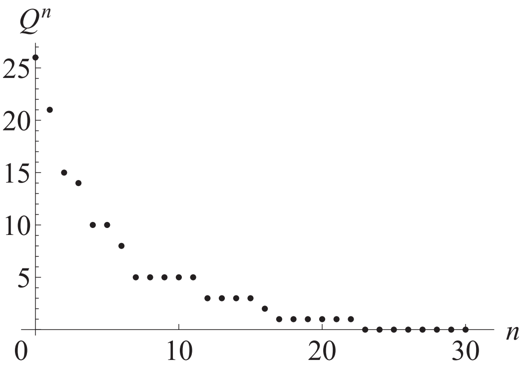


Fig. 3. Monotonical decrease of $Q^n$.

where

$$\begin{aligned} A_j^n &= \{u_{j-3}^n + 2u_{j-2}^n + 3u_{j-1}^n + u_j^n, \dots, \\ &\quad 5u_{j-1}^n + 2u_j^n\}, \\ B_j^n &= \{u_{j-3}^n + u_{j-2}^n + 4u_{j-1}^n + u_j^n, \dots, \\ &\quad u_{j-2}^n + 5u_{j-1}^n + u_j^n\}. \end{aligned}$$

We abbreviate the full form of $A_j^n$ and $B_j^n$, since they are the set of 134 and 35 elements of linear combination of $u$'s, respectively. In the derivation, we use the following principles of deformation.

- Any min function can be rewritten by max function through the formula,

$$\min(\alpha, \beta, \dots) = -\max(-\alpha, -\beta, \dots).$$

- The sum of max functions can be unified as a simple max function by the formula,

$$\begin{aligned} &\max(\alpha_1, \alpha_2, \dots) + \max(\beta_1, \beta_2, \dots) \\ &= \max(\alpha_1 + \beta_1, \alpha_1 + \beta_2, \dots, \\ &\quad \alpha_2 + \beta_1, \alpha_2 + \beta_2, \dots, \\ &\quad \dots). \end{aligned}$$

- Using the periodic boundary condition on $u_j^n$, we can shift the space variable $j$ arbitrarily for any sum of quantity $p$ defined by $u_{j+j_0}^n, u_{j+j_0+1}^n, \dots, u_{j+j_1}^n$.

$$\sum_j p(u_{j+j_0}^n, u_{j+j_0+1}^n, \dots, u_{j+j_1}^n) = \sum_j p(u_{j+j_0+k}^n, u_{j+j_0+k+1}^n, \dots, u_{j+j_1+k}^n).$$

Since $\max A_j^n \cup B_j^n - \max A_j^n \geq 0$ in RHS of (3), $Q^n - Q^{n+1} \geq 0$ always holds and $Q^n$ decreases monotonically. Fig. 3 shows an example of evolution of $Q^n$.

### *3.3 Property (III)*

Assume that $L$ is the maximum value of initial data. As previously shown, it is also the upper bound of the solution. Consider the following four kinds of local patterns,

$$LL, \quad \ell_1\ell_2, \quad \ell_1 L \ell_2, \quad LL\ell,$$

where $\ell_1$, $\ell_2$, and $\ell$ denote arbitrary values less than $L$. For the above local patterns at time $n$, we assume the values at next time $n+1$ as follows,

$$\begin{array}{ccccccc} n & : & LL & \ell_1\ell_2 & \ell_1 L\ell_2 & LL\ell \\ n+1 & : & a\,b & *\,c & *\,d\,* & a\,b\,e \end{array},$$

where the symbols $a \sim e$ and $*$ denote values at $n+1$ and especially $*$ is arbitrary. Using (2), we have

$$\begin{aligned} a &= L + \max(0, \min(u_{j-1}^n - u_{j-2}^n, \underbrace{u_{j-1}^n - L}_{\leq 0})) \\ &\quad - \max(0, \min(L - u_{j-1}^n, \underbrace{L - L}_{=0})) \\ &= L. \end{aligned}$$

By the similar discussion, we obtain $b = L$. Moreover, we have

$$\begin{aligned} c &= \ell_2 + \max(0, \min(\ell_1 - u_{j-2}^n, \ell_1 - \ell_2)) \\ &\quad - \max(0, \min(\ell_2 - \ell_1, \ell_2 - u_{j+1}^n)) \\ &= \begin{cases} \ell_2 - \max(0, \min(\ell_2 - \ell_1, \ell_2 - u_{j+1}^n)) & (\ell_1 \leq \ell_2) \\ \ell_2 + \max(0, \min(\ell_1 - u_{j-2}^n, \ell_1 - \ell_2)) & (\ell_1 > \ell_2) \end{cases} \\ &= \begin{cases} \min(\ell_2, \max(\ell_1, u_{j+1}^n)) & (\ell_1 \leq \ell_2) \\ \max(\ell_2, \min(\ell_1 + \ell_2 - u_{j-2}^n, \ell_1)) & (\ell_1 > \ell_2) \end{cases} \\ &\leq \begin{cases} \ell_2 & (\ell_1 \leq \ell_2) \\ \max(\ell_1, \ell_2) = \ell_1 & (\ell_1 > \ell_2) \end{cases} \\ &< L, \end{aligned}$$

since $\ell_1$ and $\ell_2$ is less than $L$. By the similar discussion, we obtain $d < L$ and $e = \ell$. Therefore, we get the following results,

$$\begin{array}{ccccccc} n & : & LL & \ell_1\ell_2 & \ell_1 L\ell_2 & LL\ell \\ n+1 & : & LL & *\,\ell_3 & *\,\ell_3\,* & LL\ell \end{array},$$

where $\ell_3$ denotes a value less than $L$. Then we can see the following evolutions are not allowed,

$$\begin{array}{cccccc} n & : & L\ell_1* & \ell_1 L* & \ell_1\ell_2* \\ n+1 & : & LL\ell & LL\ell & LL\ell \end{array}, \tag{4}$$

since we have

$$\begin{array}{cccccccc} n & : & L\underline{L\ell_1*} & \ell_2\underline{L\ell_1*} & \underline{\ell_1 LL} & \underline{\ell_1 L\ell_2} & \underline{\ell_1\ell_2*} \\ n+1 & : & *\underline{*\,\ell_1*} & *\underline{\ell_3\,*\,*} & \underline{*\,*\,L} & \underline{*\,\ell_3*} & \underline{*\,\ell_3*} \end{array},$$

from three kinds of pattern at time $n$ of (4). Therefore, the local patterns $LL\ell$ are stationary and those at time $n$ and $n+1$ are in one-to-one correspondence. The number of $LL\ell$ ($\#LL\ell$) is a conserved quantity of higher order for Max4-4 defined by the pattern of three successive site values.

Note that the pattern $LL\ell$ becomes 110 in the binary case $u \in \{0, 1\}$. We can consider $\#LL\ell$ of Max4-4 is a counterpart of #110 of PCA4-4.

## 4. Conclusion

We propose Max4-4 defined by the max-plus equation (2) as an extension of PCA4-4 which is a binary CA defined by (1). Max4-4 is equivalent to PCA4-4 if state values are restricted to binary values $\{0, 1\}$. Moreover, PCA4-4 has two conserved quantities shown by properties (I) and (III), and one monotonically decreasing quantity by property (II). We show that Max4-4 also has counterparts of these properties as shown in Table 1.

Table 1. Corresponding properties.

| property | (I) | (II) | (III) |
|---|---|---|---|
| PCA4-4 | #1 | $\#010^n$ | #110 |
| Max4-4 | $\sum_j u_j^n$ | $\sum_j q(u_{j-1}^n, u_j^n, u_{j+1}^n)$ | $\#LL\ell$ |

Many different max-plus equations can be found only if they include the original binary rule of PCA4-4 as a special case. However, very few among those equations may also have all max-plus counterparts of properties of PCA4-4, and we have succeeded to find one defined by (2). Moreover, about property (III), $\#LLl$ of Max4-4 is defined by the global value $\max_j\{u_j^0\}$ which is given by the total site values. On the other hand, the conserved quantity #110 of PCA4-4 is both local and global since it is obtained by the number of a pattern of successive three sites and '1' of #110 is the maximum of total site values at the same time. For Max4-4 extended from PCA4-4, we can consider that the global property of #110 is chosen and it becomes $\#LLl$. Such a relation is quite novel and it implies an interesting relevance between binary CA and max-plus equation.

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