NOLTA, IEICE

Paper

On fundamental diagram of stochastic cellular automata with a quadratic conserved quantity

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Received November 4, 2015; Revised February 29, 2016; Published July 1, 2016

Abstract: We report the exact analysis of asymptotic behavior for some stochastic cellular automata with a quadratic conserved quantity. There exists a reduction from the cellular automaton with a quadratic conserved quantity to that with a primary through a transformation of variable. Exact analysis about the asymptotic behavior is made utilizing this relation.

Key Words: cellular automaton, fundamental diagram, conserved quantity

1. Introduction

1.1 Cellular automaton defined by transportation

Cellular automaton (CA) has been proposed as a simple mathematical model of various phenomena [1]. Both its space variable, 'site', and time variable are discrete. A state variable takes a value in a finite set. Constructing a rule that determines its time evolution is simple, flexible and diverse. Due to these properties, introducing random variables to the model is easy. Stochastic CAs as well as deterministic CAs have been vastly studied and applied to physical phenomena such as combustion [2] and crystal growth [3], and advective phenomena such as fluid flow and network transmission [4, 5].

In this article, we investigate CAs which model transportation of objects such as mass and information. When we analyze transportation phenomena, the concept of conservation law is important. If there is no creation or annihilation of objects in the process of transportation, conservation law holds [6].

Another important concept of transportation phenomena is a phase transition which is observed in the case of objects with excluded volume interactions. In the field of traffic flow theory, it is well known that the transition from free-flow phase to congestion phase occurs as cars density increase. To analyze the phase transition, a 'fundamental diagram' (FD) that describes a relation between the density of objects and the mean flux over space (and time) is useful [7].

Nonlinear Theory and Its Applications, IEICE, vol. 7, no. 3, pp. 313–323 DOI: 10.1587/nolta.7.313

As a model of transportation phenomena, a binary CA (CA of which state value is 1 or 0) that conserves the number of sites of state value 1 (or 0), which we call 'particle CA', is often used and its dynamics have been studied extensively using FD [8, 9]. Various stochastic particle CAs have also been proposed in order to take account of the effect of fluctuation of transportation, and studied from theoretical viewpoint of statistical mechanics [10].

1.2 Stochastic extension of ECA184 and its fundamental diagram

We consider a 1+1 dimensional stochastic binary CA of a finite number of neighbors with conserved quantities of which FD is independent of initial data and is determined uniquely by the density and a parameter by which a random variable is characterised. In statistical mechanics of transportation phenomena, an analytic expression of FD which depends on density and parameters is useful for analyzing global asymptotic flow.

We show an example of CAs that is constructed from 'elementary cellular automaton' (ECA) of rule number 184 by introducing a random variable preserving its conserved quantity. We call the stochastic system 'SECA184' shortly. The evolution rule of SECA184 is defined by

$$u_j^{n+1} = u_j^n + q_{j-1}^n - q_j^n, \qquad q_j^n = \min(a_j^n, u_j^n, 1 - u_{j+1}^n), \tag{1}$$

where j is an integer site number, n an integer time, u_j^n and a_j^n binary state variables that take value 0 or 1. The parameter a_j^n is stochastic defined for every j and n independently by

$$a_j^n = \begin{cases} 1 & \text{(probability } \alpha) \\ 0 & (1-\alpha) \end{cases}, \tag{2}$$

where α is constant satisfying $0 \leq \alpha \leq 1$. Figure 1 shows an example of a time evolution in the case of $\alpha = 0.65$. We assume the periodic boundary condition for space sites and K denotes the number of sites. Mean flux over the space sites in the limit of $n \to \infty$ is defined by

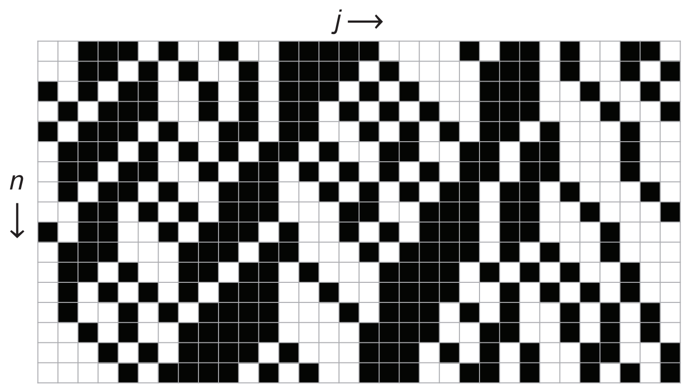

Fig. 1. Example of time evolution of SECA184 with a periodic boundary condition. Symbols ■ and □ denote $u = 1$ and 0 respectively.

$$Q = \lim_{n\to\infty} \frac{1}{K} \sum_{j=1}^{K} q_j^n. \tag{3}$$

Figure 2 shows that Q is determined uniquely by ρ_1 (density of u) and α, and is independent of initial data. The analytic expression of this FD was first given by Schadschneider and Schreckenberg as follows [10]:

$$Q = \frac{1 - \sqrt{1 - 4\alpha\rho_1(1-\rho_1)}}{2}. \tag{4}$$

1.3 ECAs with a quadratic conserved quantity

It is known that there are ECAs with a conserved quantity of higher order [6, 11–14]. If an average of n-th order quantity of state value u is conserved, we call it the conserved quantity of order n. For

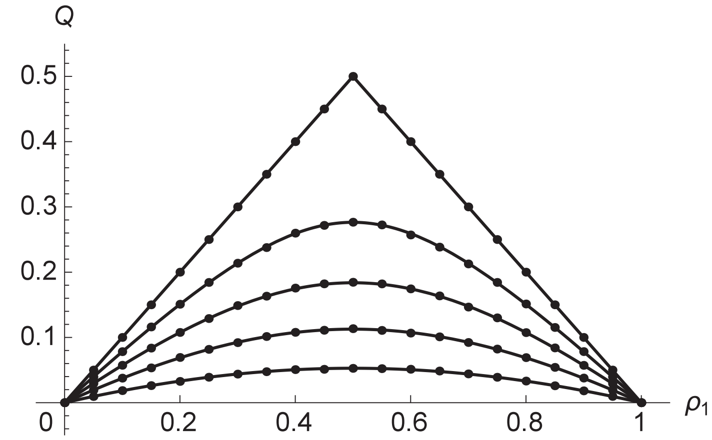

Fig. 2. Numerical results (dots) and theoretical curves of FD of SECA184 for $\alpha = 0.2, 0.4, 0.6, 0.8, 1$.

example, ECA212 and ECA84 have a conserved quantity which is the sum of quadratic density of pattern 01 and 10. For later discussions, we use the notation $\rho_{x_1 x_2 \dots x_k}$ which means the density of pattern $x_1 x_2 \dots x_k$. We discuss two evolutional systems relating each other through a transformation and treat densities of both systems. To avoid confusion, we use symbols ρ and σ denoting that of each system.

1.3.1 ECA212

Time evolution equation of ECA212 is given by

$$v_j^{n+1} = v_j^n \oplus \min(v_{j-1}^n \oplus v_j^n, 1 - v_j^n \oplus v_{j+1}^n). \tag{5}$$

Note that we use the symbol v instead of u denoting state value and the operator $\oplus$ is a binary operation defined by

$$x \oplus y = x + y \mod 2 \qquad (x, y \in \{0, 1\}). \tag{6}$$

We have the following formulas for $x \in \{0, 1\}$,

$$x \oplus 0 = 0 \oplus x = x, \qquad x \oplus 1 = 1 \oplus x = 1 - x, \qquad x \oplus x = 0. \tag{7}$$

The evolution rule (5) of ECA212, which is equivalent to ECA142 through the reflection of space, can be written in the following transition table [12].

$v_{j-1}^n v_j^n v_{j+1}^n$	111	110	101	100	011	010	001	000
v_j^{n+1}	1	1	0	1	0	1	0	0

(8)

An example of time evolution of ECA212 is shown in Fig. 3.

Considering the transformation

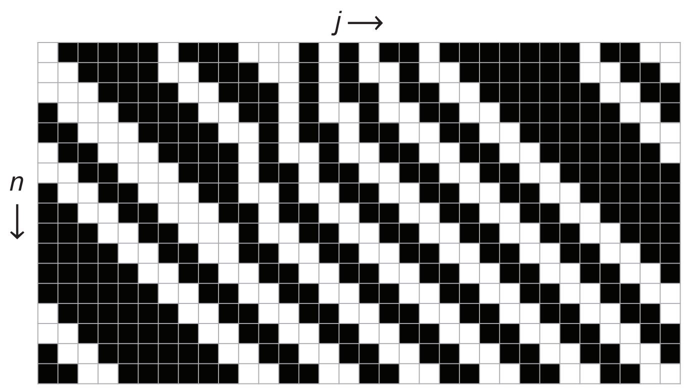

Fig. 3. Example of time evolution of ECA212.

$$u_j^n = v_{j-1}^n \oplus v_j^n, \tag{9}$$

we obtain the following time evolution equation on u from (5),

$$u_j^{n+1} = u_j^n \oplus \min(u_{j-1}^n, 1-u_j^n) \oplus \min(u_j^n, 1-u_{j+1}^n). \tag{10}$$

If we replace the first and the second $\oplus$ of (10) by $+$ and $-$ respectively, we obtain

$$u_j^{n+1} = u_j^n + \min(u_{j-1}^n, 1-u_j^n) - \min(u_j^n, 1-u_{j+1}^n). \tag{11}$$

Since RHS's of (10) and (11) give the same value if u is restricted to $u \in \{0,1\}$, we can use (11) in place of (10).

Equation (11) (or (10)) is nothing but ECA184, and it has a conserved quantity ρ_1. Therefore, ECA212 defined by (5) or (8) has the following quadratic conserved quantity through (9) [12–14].

$$\sigma_{01} + \sigma_{10} = \frac{1}{K}\sum_{j=1}^{K} v_j^n \oplus v_{j+1}^n. \tag{12}$$

To avoid the confusion, we use the symbols ρ and σ for the density of u and v respectively. Note that $\sigma_{01} = \sigma_{10}$ and both σ_{01} and σ_{10} are conserved for the periodic boundary condition or the boundary condition $\lim_{|j|\to\infty} u_j^n = 0$ in the infinite domain. Hereafter, we assume the condition $\sigma_{01} = \sigma_{10}$.

1.3.2 ECA84

The transition table of ECA84 is given by the following table.

$v_{j-1}^n v_j^n v_{j+1}^n$	111	110	101	100	011	010	001	000
v_j^{n+1}	0	1	0	1	0	1	0	0

(13)

We rewrite this table in the form of 4 neighbors.

$v_{j-1}^n v_j^n v_{j+1}^n v_{j+2}^n$	1111	1110	1101	1100	1011	1010	1001	1000
$v_j^{n+1} v_{j+1}^{n+1}$	00	01	10	11	00	01	10	10

0111	0110	0101	0100	0011	0010	0001	0000
00	01	10	11	00	01	00	00

(14)

We can easily show

$$\sigma_{01}^{n+1} = \sigma_{1110}^n + \sigma_{1010}^n + \sigma_{0110}^n + \sigma_{0010}^n = \sigma_{10}^n, \tag{15}$$
$$\sigma_{10}^{n+1} = \sigma_{1101}^n + \sigma_{1001}^n + \sigma_{1000}^n + \sigma_{0101}^n = \sigma_{10}^n, \tag{16}$$

from this table, and have

$$\sigma_{01}^{n+1} = \sigma_{10}^{n+1} = \sigma_{10}^n. \tag{17}$$

Thus σ_{10}^n and σ_{01}^n are both conserved and equivalent to each other.

We can express the transition table (13) by the equation,

$$v_j^{n+1} = v_j^n \oplus \min(v_{j-1}^n \oplus v_j^n + v_{j+1}^n, 1 - v_j^n \oplus v_{j+1}^n). \tag{18}$$

An example of time evolution of ECA84 is shown in Fig. 4.

If we consider the same transformation as (9),

$$u_j^n = v_{j-1}^n \oplus v_j^n, \tag{19}$$

to (18), we can derive the evolution equation for some specific boundary conditions. Consider the infinite space and the boundary condition $\lim_{|j|\to\infty} v_j^n = 0$, we have

$$u_j^{n+1} = u_j^n \oplus \min\Big(u_{j-1}^n + \bigoplus_{i=j+1}^{\infty} u_i^n, 1-u_j^n\Big) \oplus \min\Big(u_j^n + \bigoplus_{i=j+2}^{\infty} u_i^n, 1-u_{j+1}^n\Big), \tag{20}$$

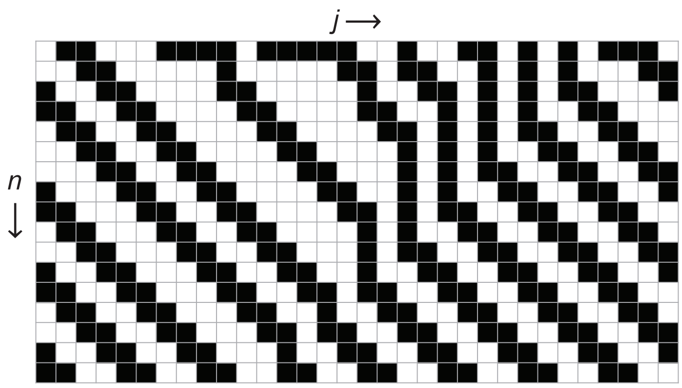

Fig. 4. Example of time evolution of ECA84.

where $\oplus_{i=k_1}^{k_2} f_i$ means

$$\bigoplus_{i=k_1}^{k_2} f_i = f_{k_1} \oplus f_{k_1+1} \oplus \cdots \oplus f_{k_2}. \tag{21}$$

Furthermore, (20) can be rewritten by conservation form

$$u_j^{n+1} = u_j^n + \min\Big(u_{j-1}^n + \bigoplus_{i=j+1}^{\infty} u_i^n, 1 - u_j^n\Big) - \min\Big(u_j^n + \bigoplus_{i=j+2}^{\infty} u_i^n, 1 - u_{j+1}^n\Big), \tag{22}$$

for the binary value $u \in \{0, 1\}$. This form means that ρ_1 is conserved.

However, if we consider the periodic boundary for finite space sites, the inverse transformation of (19) becomes indefinite. To avoid this difficulty, we use the following notation to express the solution on u. The densities σ_{01}^n and σ_{10}^n are both conserved for ECA84 and they are equal to each other. Using the transformation (19), this quantity becomes the density of the sites $u = 1$. Therefore, we can regard the sites of $u = 1$ as moving particles. Let us distinguish two kinds of particle $u_j^n = 1$ corresponding to $(v_{j-1}^n, v_j^n) = (0, 1)$ and $(v_{j-1}^n, v_j^n) = (0, 1)$ respectively. We define the former by $u_j^n = L$ and the latter by $u_j^n = R$. The following is an example of this notation.

$$\begin{aligned} v_j^n &: \ \ldots \texttt{00011110001010011000} \ldots \\ u_j^n &: \ \ldots \texttt{000L000R00LRLR0L0R00} \ldots \end{aligned} \tag{23}$$

The symbols L and R mean the left and the right ends of sequence $11 \ldots 1$ of v respectively.

The time evolution rule of u is expressed by the motion rule of L and R as Lagrangian expression.

- If L is adjacent to the left of R, it does not move. Otherwise it moves to the adjacent left site to its nearest right R.
- If R at site j is adjacent to the left of L, it does not move. Otherwise it moves to its nearest right site.

Note that this rule can be applied to the periodic boundary condition.

In this article, we propose stochastic extensions of ECA212 and ECA84 introducing the stochastic parameter preserving their quadratic conserved quantity. We call the stochastic extensions SECA212 and SECA84 respectively. Moreover we derive the analytic expressions of FD for SECA212 and SECA84.

2. Analytic expression of FD of SECA212

We obtain SECA184 (1) by introducing the random variable preserving the conserved quantity. Using this fact, ECA212 can be randomized through the transformation (9). The process of this randomization is as follows:

$$\text{SECA184: } u_j^{n+1} = u_j^n + \min(a_{j-1}^n, u_{j-1}^n, 1 - u_j^n) - \min(a_j^n, u_j^n, 1 - u_{j+1}^n)$$

$$\updownarrow \quad \text{(equivalent as a binary system)}$$

$$\begin{aligned} \text{SECA184: } u_j^{n+1} &= u_j^n \oplus \min(a_{j-1}^n, u_{j-1}^n, 1-u_j^n) \oplus \min(a_j^n, u_j^n, 1-u_{j+1}^n) \\ \updownarrow \quad u_j^n &= v_{j-1}^n \oplus v_j^n \quad \text{(transformation)} \\ \text{SECA212: } v_j^{n+1} &= v_j^n \oplus \min(a_j^n, v_{j-1}^n \oplus v_j^n, 1 - v_j^n \oplus v_{j+1}^n) \end{aligned} \tag{24}$$

where a_j^n is stochastic defined by

$$a_j^n = \begin{cases} 1 & \text{(probability } \alpha) \\ 0 & (1-\alpha) \end{cases} . \tag{25}$$

The process shows clearly that $\sigma_{01} + \sigma_{10}$ is conserved for SECA212. The transition table of SECA212 is as follows.

$v_{j-1}^n v_j^n v_{j+1}^n$	111	110	101	100	011	010	001	000
v_j^{n+1}	1	1	0	a_j^n	$1-a_j^n$	1	0	0

(26)

An example of time evolution of SECA212 is shown in Fig. 5. The mean flux of SECA184 reduces to

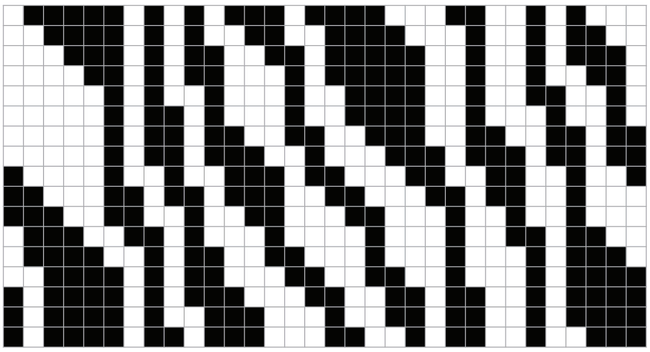

Fig. 5. Example of time evolution of SECA212 for $\alpha = 0.6$.

$$Q = \lim_{n\to\infty} \frac{1}{K} \sum_{j=1}^{K} \min(a_j^n, v_{j-1}^n \oplus v_j^n, 1 - v_j^n \oplus v_{j+1}^n), \tag{27}$$

for SECA212 by the transformation. If $u_j^n = 1$, (v_{j-1}^n, v_j^n) is equal to $(0,1)$ or $(1,0)$. Thus Q shows the mean flux of borders between sequences of one or more 0's and 1's in SECA212. FD of SECA212 is easily obtained by replacing the horizontal axis of FD of SECA184 of Fig. 2 by $2\sigma_{01}$ $(= \sigma_{01} + \sigma_{10})$ and the vertical axis by Q in (27). Thus the analytic expression of FD of SECA212 is

$$Q = \frac{1 - \sqrt{1 - 8\alpha\sigma_{01}(1 - 2\sigma_{01})}}{2}. \tag{28}$$

3. Analytic expression of FD of SECA84

3.1 Definition of SECA84

We can introduce an external variable ($a_j^n \in \{0,1\}$) into ECA84 (18) preserving the conserved quantity σ_{01} $(= \sigma_{10})$. The evolution equation of SECA84 is defined by

$$v_j^{n+1} = v_j^n \oplus \min(\min(a_j^n, v_{j-1}^n \oplus v_j^n) + v_{j+1}^n, 1 - v_j^n \oplus v_{j+1}^n), \tag{29}$$

where

$$a_j^n = \begin{cases} 1 & \text{(probability } \alpha) \\ 0 & (1-\alpha) \end{cases} . \tag{30}$$

The evolution rule of this equation is expressed by the following equivalent binary transition table.

$v_{j-1}^n v_j^n v_{j+1}^n$	111	110	101	100	011	010	001	000
v_j^{n+1}	0	1	0	a_j^n	0	1	0	0

(31)

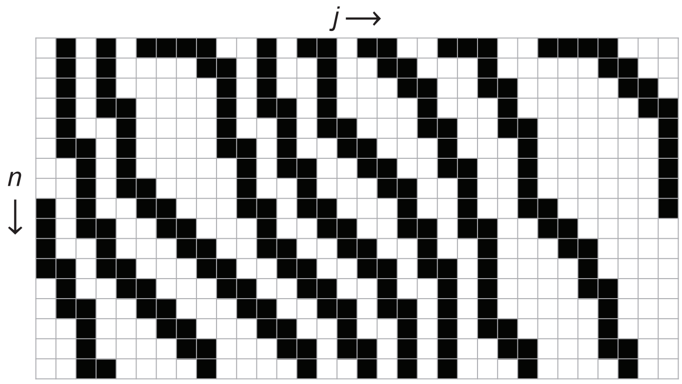

Fig. 6. Example of time evolution of SECA84 for $\alpha = 0.6$.

An example of time evolution of SECA84 is shown in Fig. 6. We can express the motion rule of both ends of sequences $11\dots1$ by L and R particles as ECA84 in the previous section. The motion rule is as follows:

- If L is adjacent to the left of R, it does not move. Otherwise it moves to the adjacent left site to its nearest right R.
- If R at site j is adjacent to the left of L, it does not move. Otherwise it moves to its nearest right site if $a_j^n = 1$ and does not $a_j^n = 0$.

The flux q_j^n is defined by the momentum of L and R particles,

$$q_j^n = \min(\min(a_j^n, v_{j-1}^n \oplus v_j^n) + v_{j+1}^n, 1 - v_j^n \oplus v_{j+1}^n), \tag{32}$$

of which transition table is as follows.

$v_{j-1}^n v_j^n v_{j+1}^n$	111	110	101	100	011	010	001	000
q_j^n	1	0	0	a_j^n	1	0	0	0

(33)

The following is a schematic example showing the relation among v, L, R and q.

$$\begin{aligned} v_j^n &: \ \dots\texttt{00011110001010011000}\dots \\ u_j^n &: \ \dots\texttt{000L000R00LRLR0L0R00}\dots \\ q_j^n &: \ \dots\texttt{0001110a00000a010a00}\dots \end{aligned} \tag{34}$$

Finally the mean flux of SECA84 is defined by

$$Q = \lim_{n\to\infty} \frac{1}{K} \sum_{j=1}^{K} q_j^n. \tag{35}$$

3.2 FD of SECA84

FD of SECA84 is a relation between the quadratic conserved quantity of v and the mean flux Q. This relation can be transformed to that of the L and R particle system through the transformation $u_j^n = v_{j-1}^n \oplus v_j^n$. Assume that ρ_1 denotes the density of L and R particle. It is equal to the conserved quantity $\sigma_{01} + \sigma_{10} = 2\sigma_{01}$. Thus FD of SECA84 can be considered as a relation between ρ_1 and Q.

Let us define a new notation $\rho^n_{x_1 x_2 \dots x_k}$ denoting the density of the pattern $x_1 x_2 \dots x_k$ in L and R particle system at time n where each x_i is 0, L or R. Then ρ_L^n and ρ_R^n satisfy the following condition,

$$\rho_L^n = \rho_R^n = \frac{\rho_1}{2}. \tag{36}$$

Thus ρ_L^n and ρ_R^n are conserved and we rewrite them as ρ_L and ρ_R removing the superscript n. Moreover

$$\rho^n_{0\dots0L} = \rho^n_{R0\dots0}, \tag{37}$$

holds since the position of L's and R's is alternate and we have

$$\rho^n_{0\dots0L} = \rho^n_{00\dots0L} + \rho^n_{R0\dots0L}, \qquad \rho^n_{R0\dots0} = \rho^n_{R0\dots00} + \rho^n_{R0\dots0L}, \tag{38}$$

thus

$$\rho^n_{00\dots0L} - \rho^n_{R0\dots00} = \rho^n_{0\dots0L} - \rho^n_{R0\dots0} = \cdots = \rho^n_{0L} - \rho^n_{R0} = \rho_L - \rho_R = 0. \tag{39}$$

Note that the length of sequences $11\dots1$ of v is always equal to 1 or 2 for $n \geq 1$ according to the evolution rule (29). Thus there exist seven configurations 00, $0L$, $0R$, $L0$, LR, $R0$, RL about the neighboring two sites in the L and R particle system for $n \geq 1$. Those densities are expressed by ρ_{0R}, ρ_{R0} and ρ_1 as follows.

$$\begin{aligned}
\rho^n_{0L} &= \rho^n_{R0}, \\
\rho^n_{L0} &= \rho^n_{L0R} = \rho^n_{0R}, \\
\rho^n_{LR} &= \rho_L - \rho^n_{L0} = \frac{\rho_1}{2} - \rho^n_{0R}, \\
\rho^n_{RL} &= \rho_R - \rho^n_{R0} = \frac{\rho_1}{2} - \rho^n_{R0}, \\
\rho^n_{00} &= 1 - (\rho^n_{0R} + \rho^n_{0L} + \rho^n_{R0} + \rho^n_{L0} + \rho^n_{LR} + \rho^n_{RL}) = 1 - \rho_1 - \rho^n_{0R} - \rho^n_{R0}.
\end{aligned} \tag{40}$$

Assume that average density of $\rho^n_{x_1x_2\dots x_k}$ over n converges to a constant $\rho_{x_1x_2\dots x_k}$ for $n \to \infty$. Since the configuration $(u^{n+1}_j, u^{n+1}_{j+1}) = (0, R)$ is obtained iff $(u^n_j, u^n_{j+1}) = (R, 0)$ and $a^n_j = 1$, we have

$$\rho_{0R} = \alpha\rho_{R0}. \tag{41}$$

The particle L moves to the right by one site iff $(u^n_j, u^n_{j+1}) = (L, 0)$ and R moves to the right by one site iff $(u^n_j, u^n_{j+1}) = (R, 0)$ and $a^n_j = 1$. Therefore we obtain

$$Q = \rho_{L0} + \alpha\rho_{R0} = 2\alpha\rho_{R0}. \tag{42}$$

Thus we express ρ_{R0} by ρ_1 and α to derive the analytic expression of FD.

For this purpose, we derive the closed form of system of equations on ρ_{R0}, ρ_1 and α using equations derived by motion of particles and reduction relations about densities. The first equation is obtained by the transition probability of distribution of u at time n which gives $(u^{n+1}_j, u^{n+1}_{j+1}) = (R, 0)$ at time $n+1$.

u^n_{j-1}	u^n_j	u^n_{j+1}	u^n_{j+2}	transition probability
	R	0		$1-\alpha$
R	0	0		α
R	0	L	0	α
	R	L	0	1

(43)

From these distributions, we have

$$\rho_{R0} = (1-\alpha)\rho_{R0} + \alpha(\rho_{R00} + \rho_{R0L0}) + \rho_{RL0}. \tag{44}$$

The second equation is obtained considering the distributions giving $(u^{n+1}_{j-1}, u^{n+1}_j, u^{n+1}_{j+1}) = (0, 0, 0)$.

u^n_{j-2}	u^n_{j-1}	u^n_j	u^n_{j+1}	u^n_{j+2}	transition probability
0	0	0	0		1
R	0	0	0		$1-\alpha$
R	0	0	L	0	$1-\alpha$
0	0	0	L	0	1

(45)

Then we obtain

$$\rho_{000} = \rho_{0000} + (1-\alpha)(\rho_{R000} + \rho_{R00L0}) + \rho_{000L0}. \tag{46}$$

Since there are 13 kinds of $\rho^n_{x_1x_2x_3}$ for $n \geq 1$, that is,

$$\begin{array}{l} \rho^n_{000},\ \rho^n_{00L},\ \rho^n_{0L0},\ \rho^n_{0LR},\ \rho^n_{0R0},\ \rho^n_{0RL}, \\ \rho^n_{L0R},\ \rho^n_{LR0},\ \rho^n_{LRL},\ \rho^n_{R00},\ \rho^n_{R0L},\ \rho^n_{RL0},\ \rho^n_{RLR}, \end{array} \tag{47}$$

the following equation holds.

$$\begin{aligned} \rho^n_{000} = 1 - (&\rho^n_{00L} + \rho^n_{0L0} + \rho^n_{0LR} + \rho^n_{0R0} + \rho^n_{0RL} \\ &+\rho^n_{L0R} + \rho^n_{LR0} + \rho^n_{LRL} + \rho^n_{R00} + \rho^n_{R0L} + \rho^n_{RL0} + \rho^n_{RLR}). \end{aligned} \tag{48}$$

By the definition of L and R, we always have

$$\rho^n_{R0L} = \rho^n_{R0} - \rho^n_{R00}. \tag{49}$$

One of the direct consequences of (37) is

$$\rho^n_{00L} = \rho^n_{R00}. \tag{50}$$

Moreover, the equation

$$\rho^n_{L0R} = \rho^n_{0R}, \tag{51}$$

holds for $n \geq 1$ from the motion rule of L and R.

The accurate numerical results support the reduction relation

$$\rho_{x_1x_2x_3} = \frac{\rho_{x_1x_2}\rho_{x_2x_3}}{\rho_{x_2}} \tag{52}$$

for $x_1x_2x_2 = 0L0$, $0LR$, $0R0$, $0RL$, $LR0$, LRL, $RL0$, RLR. This kind of relation among densities is called 'n-cluster approximation' [15, 16]. Using these reduction relations together with (48), (49), (50) and (51), we can express $\rho_{x_1x_2x_3}$ by ρ_{x_1}, $\rho_{x_1x_2}$ and ρ_{R00}.

Furthermore, we confirm the following reduction relations hold for any combination of x_i by the numerical calculations.

$$\rho_{x_1x_2x_3x_4} = \frac{\rho_{x_1x_2x_3}\rho_{x_2x_3x_4}}{\rho_{x_2x_3}}, \qquad \rho_{x_1x_2x_3x_4x_5} = \frac{\rho_{x_1x_2x_3x_4}\rho_{x_2x_3x_4x_5}}{\rho_{x_2x_3x_4}}. \tag{53}$$

Thus $\rho_{x_1x_2x_3x_4}$ and $\rho_{x_1x_2x_3x_4x_5}$ can also be expressed by ρ_{x_1}, $\rho_{x_1x_2}$ and ρ_{R00}. Substituting the above relations into (44) and (46) and eliminating ρ_{R00}, we obtain

$$2\alpha(\rho_{R0})^2 - \Big(1 - (1-\alpha)\frac{\rho_1}{2}\Big)\rho_{R0} + \frac{1}{2}\rho_1(1-\rho_1) = 0. \tag{54}$$

Solving this quadratic equation,

$$\rho_{R0} = \frac{1}{4\alpha}\Big\{1 - (1-\alpha)\frac{\rho_1}{2} - \sqrt{\Big(1-(1-\alpha)\frac{\rho_1}{2}\Big)^2 - 4\alpha\rho_1(1-\rho_1)}\Big\} \tag{55}$$

is derived. Therefore, the mean flux becomes

$$Q = 2\alpha\rho_{R0} = \frac{1}{2}\Big\{1 - (1-\alpha)\frac{\rho_1}{2} - \sqrt{\Big(1-(1-\alpha)\frac{\rho_1}{2}\Big)^2 - 4\alpha\rho_1(1-\rho_1)}\Big\}, \tag{56}$$

from (42). Figure 7 shows comparison between the analytic expression and the numerical result. We can observe that both coincide well. If we consider the variable of horizontal axis is $\sigma_{01} + \sigma_{10}$, this FD is nothing but that of SECA84.

4. Concluding remarks

In this article, we introduce random variables to deterministic systems preserving their quadratic conserved quantity, and propose the method to derive the analytic expression of FD giving the relation between conserved quantity and mean flux. We utilize the variable transformation between the system with the quadratic conserved quantity and that with the primary. In the derivation, we use three assumptions as follows:

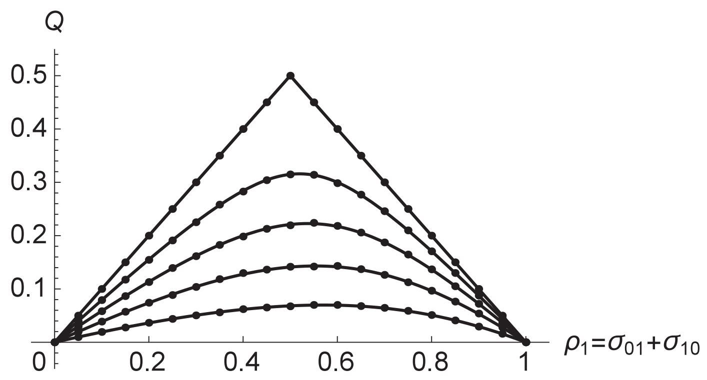

Fig. 7. FD of (29). Solid lines are drawn by analytic expression (56) and dots numerical results.

- If conserved quantity and stochastic parameter are given, average density of $\rho^n_{x_1 \dots x_k}$ converges to a constant value for $n \to \infty$.
- Reduction relation $\rho_{x_1 x_2 \dots x_{k-1} x_k} = \rho_{x_1 x_2 \dots x_{k-1}} \rho_{x_2 \dots x_{k-1} x_k} / \rho_{x_2 \dots x_{k-1}}$ holds for some patterns of $(x_1, \dots, x_k)$.
- FD is determined uniquely by conserved quantity and stochastic parameter. It does not depend on initial distributions of 0 and 1.

We make numerical verification with some level of accuracy about the above assumptions. It is one of future problems to prove them exactly. There are other deterministic or stochastic systems with conserved quantity, but some of above assumptions are not satisfied generally. Therefore the assumptions are strong requirement for the dynamics of system and the conditions to support the assumptions are necessary to be clarified.

Moreover there are various systems of multi-valued or multi-neighbors with conserved quantities. In the nonlinear dynamical system theory, it is important to clarify the general structure of particle systems where exact analysis of the asymptotic behavior is possible and we expect the results obtained in this article can contribute to the problem.

Acknowledgments

This work was supported by JSPS KAKENHI Grant Number 26600156 and CREST, JST.

References

[1] S. Wolfram, *A New Kind of Science*, Wolfram Media, Champaign, 2002.

[2] R. Saïd and R. Borghi, "A simulation with a cellular automaton for turbulent combustion modelling," *Proceedings of the Combustion Institute*, vol. 22, pp. 569–577, 1989.

[3] F.C. Richardsa, T.P. Meyerb, and N.H. Packardc, "Extracting cellular automaton rules directly from experimental data," *Physica D: Nonlinear Phenomena*, vol. 45, pp. 189–202, 1990.

[4] U. Frisch, B. Hasslacher, and Y. Pomeau, "Lattice-Gas Automata for the Navier-Stokes Equation," *Phys. Rev. Lett.*, vol. 56, pp. 1505–1508, 1986.

[5] K. Nagel and M. Schreckenberg, "A cellular automaton model for freeway traffic," *J. Phys. I*, vol. 2, pp. 2221–2229, 1992.

[6] T. Hattori and S. Takesue, "Additive conserved quantities in discrete-time lattice dynamical systems," *Physica D: Nonlinear Phenomena*, vol. 49, pp. 295–322, 1991.

[7] K. Nishinari and D. Takahashi, "Analytical properties of ultradiscrete Burgers equation and rule-184 cellular automaton," *J. Phys. A: Math. Gen.*, vol. 31, pp. 5439–5450, 1998.

[8] D. Takahashi, J. Matsukidaira, H. Hara, and B. Feng, "Max-plus analysis on some binary particle systems," *J. Phys. A: Math. Theor.*, vol. 44, 135102, 21 pages, 2011.

[9] T. Okumura, J. Matsukidaira, and D. Takahashi, "Max-min-plus expressions for one-dimensional particle cellular automata obtained from a fundamental diagram," *J. Phys. A: Math. Theor.*, vol. 46, 295101, 2013.

[10] A. Schadschneider and M. Schreckenberg, "Cellular automation models and traffic flow," *J. Phys. A: Math. Gen.*, vol. 26, pp. L679–L683, 1993.

[11] N. Boccara, "Transformations of one-dimensional cellular automaton rules by translation-invariant local surjective mappings," *Physica D: Nonlinear Phenomena*, vol. 68, pp. 416–426, 1993.

[12] H. Fukś, "Dynamics of the cellular automaton rule 142," *Complex Systems*, vol. 16, pp. 123–138, 2005.

[13] H. Fukś, "Second order additive invariants in elementary cellular automata," *arXiv preprint* nlin/0502037, 2005.

[14] H. Fukś and J. Haroutunian, "Catalan numbers and power laws in cellular automaton rule 14," *J. Cell. Autom.*, vol. 4, pp. 99–110, 2009.

[15] H.A. Gutowitz, J.D. Victor, and B.W. Knight, "Local structure theory for cellular automata," *Physica D*, vol. 28, pp. 18–48, 1988.

[16] M. Schreckenberg, A. Schadschneider, K. Nagel, and N. Ito, "Discrete stochastic models for traffic flow," *Phys. Rev. E*, vol. 51, pp. 2939–2949, 1995.