Traffic Congestion Models and Ultradiscretization

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Traffic Congestion Models

Mathematical models to simulate the congestion caused
by the delay of acceleration of drivers in a one-way road.
(No accident, no walker, no signal, ...)

$\rightarrow$ Interesting abstract particle models

Different level of discreteness

Example

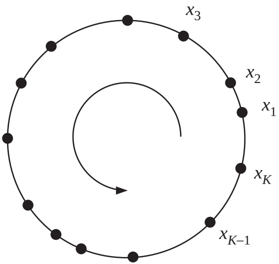
PDE:	Burgers eq, Payne model
ODE:	Optimal velocity model
CA:	Nagel–Schreckenberg model, Fukui–Ishibashi model

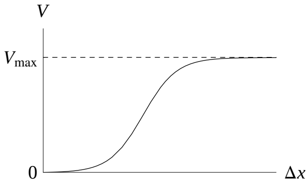
Optimal velocity (OV) model

M.Bando, K.Hasebe, A.Nakayama, A Shibata and Y.Sugiyama
"Dynamical model of traffic congestion and numerical simulation"
Phys. Rev. E, **51** (1995) 1035–1042

$$\boxed{\ddot{x}_k = A(V(x_{k+1} - x_k) - \dot{x}_k)}$$

$x_k(t)$: position of k-th car at time t
$V(\Delta x)$: optimal velocity for the distance Δx





OV demo (Numerical calculation by Runge–Kutta method)

A family of Burgers eq of different discrete levels (differential, difference, ultradiscrete)

K.Nishinari and D.Takahashi, "Analytical properties of ultradiscrete Burgers equation and rule–184 cellular automaton" J. Phys. A, **31** (1998) 5439–5450

Burgers eq	transformation	diffusion eq
$u_t = 2uu_x + u_{xx}$	$u = (\log f)_x$	$f_t = f_{xx}$

$$\Uparrow \quad u_j^n = u(j\Delta x, n\Delta t), \quad f_j^n = f(j\Delta x, n\Delta t), \quad \Delta x,\ \Delta t \to 0$$

$\dfrac{u_j^{n+1}}{u_j^n} = \dfrac{u_{j+1}^n + 1/u_j^n}{u_j^n + 1/u_{j-1}^n}$	$u_j^n = \dfrac{f_{j+1}^n}{f_j^n}$	$f_j^{n+1} = \dfrac{1}{2}(f_{j+1}^n + f_{j-1}^n)$

$$\Downarrow \quad u_j^n = e^{U_j^n/\varepsilon}, \quad f_j^n = e^{F_j^n/\varepsilon}, \quad \varepsilon \to +0$$

$U_j^{n+1} - U_j^n$ $= \min(U_{j-1}^n, 1 - U_j^n)$ $- \min(U_j^n, 1 - U_{j+1}^n)$	$U_j^n = F_{j+1}^n$ $- F_j^n + \dfrac{1}{2}$	$F_j^{n+1} = \max(F_{j+1}^n, F_{j-1}^n)$

Ultradiscrete Burgers Equation

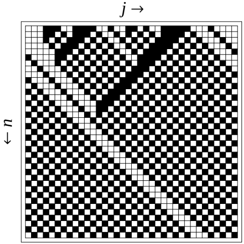
ultradiscrete Burgers eq $\ni$ rule 184 elementary CA

$$U_j^{n+1} - U_j^n = \min(U_{j-1}^n, 1 - U_j^n) - \min(U_j^n, 1 - U_{j+1}^n)$$

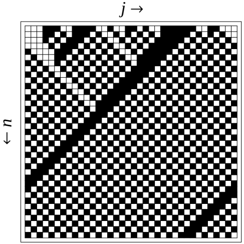
$U_{j-1}^n U_j^n U_{j+1}^n$	111	110	101	100	011	010	001	000
U_j^{n+1}	1	0	1	1	1	0	0	0

ρ: average number of cars per site

phase transition at $\rho = 1/2$



$\rho < 1/2$ (free flow)



$\rho > 1/2$ (congested flow)

Euler and Lagrange Expressions

Direct relations between known congestion models

Euler model		Lagrange model
Burgers (PDE)		???
$\downarrow$ discretize		
d-Burgers		???
$\downarrow$ ultradiscretize		
u-Burgers		???
$\cup$		
ECA184 (CA)	$\in$	Fukui–Ishibashi (FI) model
$\boxed{\begin{aligned} &U_j^{n+1} - U_j^n \\ &= \min(U_{j-1}^n, 1 - U_j^n) \\ &- \min(U_j^n, 1 - U_{j+1}^n) \end{aligned}}$		$\boxed{\begin{aligned} &x_k^{n+1} - x_k^n \\ &= \max(0, x_{k+1} - x_k - 1) \\ &- \max(0, x_{k+1} - x_k - v_{\mathsf{max}} - 1) \end{aligned}}$
		($v_{\mathsf{max}} = 1 \ \rightarrow$ ECA184)
U_j^n:number of car at site j and time n		x_k^n:position of k-th car at time n

Ultradiscretization of OV Model

D.Takahashi and J.Matsukidaira, "On a discrete optimal velocity model and its continuous and ultradiscrete relatives", JSIAM Letters **1** (2009) 1-4
J.Matsukidaira and K.Nishinari, "Euler-Lagrange Correspondence of Cellular Automaton for Traffic-Flow Models", Phys. Rev. Lett. **90** (2003) 088701

OV model is a Lagrange model of ODE type.
What is the ultradiscrete version of OV model? And how?

$$\boxed{\ddot{x}_k = A(V(x_{k+1} - x_k) - \dot{x}_k)} \quad (x_k(t)\text{:position of } k\text{-th car at time } t)$$

"Minus" problem of ultradiscretization

$$\begin{array}{llll}
z = x + y & \rightarrow \quad Z = \lim\limits_{\varepsilon \to +0} \varepsilon \log(e^{X/\varepsilon} + e^{Y/\varepsilon}) & \rightarrow & Z = \max(X, Y) \\
z = x * y & \rightarrow \quad Z = (\lim\limits_{\varepsilon \to +0}) \varepsilon \log(e^{X/\varepsilon} + e^{Y/\varepsilon}) & \rightarrow & Z = X + Y \\
z = x / y & \rightarrow \quad Z = (\lim\limits_{\varepsilon \to +0}) \varepsilon \log(e^{X/\varepsilon} / e^{Y/\varepsilon}) & \rightarrow & Z = X - Y \\
z = x - y & \rightarrow \quad Z = \lim\limits_{\varepsilon \to +0} \varepsilon \log(e^{X/\varepsilon} - e^{Y/\varepsilon}) & \rightarrow & \text{not well-defined}
\end{array}$$

$\log$ and $\exp$ are the Gateway

R.Hirota, "Nonlinear partial differential equations II: discrete-time Toda equation", J. Phys. Soc. Jpn. **43** (1977) 2074–2078
J.Matsukidaira, J.Satsuma, D.Takahashi, T.Tokihiro, M.Torii, "Toda-type cellular automaton and its N-soliton solution", Phys. Lett. A **225** (1997) 287–295

Successful example: Toda eq

differential $\boxed{\ddot{u}_k = e^{u_{k+1}} - 2e^{u_k} + e^{u_{k-1}}}$

$$\uparrow \quad \delta \to 0$$

difference $\boxed{\Delta_n^2 u_k^n = \Delta_k^2 \log(1 + \delta^2(e^{u_k^n} - 1))}$

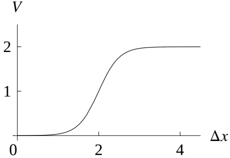
$$\downarrow \quad u_k^n \to U_k^n/\varepsilon, \delta = e^{-1/\varepsilon}, \varepsilon \to +0$$

ultradiscrete $\boxed{\Delta_n^2 U_k^n = \Delta_k^2 \max(0, U_k^n - 1)}$

Difference eq is the key!

$$\log(1 + \delta^2(e^{u_k^n} - 1)) \to \begin{cases} \log(1 + \delta^2(e^{u_k^n} - 1)) \sim \delta^2(e^{u_k^n} - 1) \\ \lim\limits_{\varepsilon \to +0} \varepsilon \log(e^{A/\varepsilon} + e^{B/\varepsilon}) = \max(A, B) \end{cases}$$

Ultradiscrete OV Model



OV $\boxed{\ddot{x}_k = A(V(x_{k+1} - x_k) - \dot{x}_k)}$

$\uparrow \quad \delta \to 0$

d-OV
$$\boxed{\begin{aligned} & x_k^{n+1} - 2x_k^n + x_k^{n-1} \\ & = A\{\log\big(1 + \delta^2 V(x_{k+1}^n - x_k^n)\big) - \log\big(1 + \delta(e^{x_k^n - x_k^{n-1}} - 1)\big)\} \end{aligned}}$$

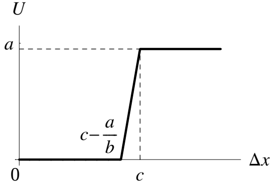
$\downarrow \quad \varepsilon \to +0$

u-OV $\boxed{x_k^{n+1} - 2x_k^n + x_k^{n-1} = A\{U(x_{k+1}^n - x_k^n) - \max(0, x_k^n - x_k^{n-1})\}}$

$$V(\Delta x) = a\left(\frac{1}{1 + e^{-b(\Delta x - c)}} - \frac{1}{1 + e^{bc}}\right)$$

$$\downarrow$$

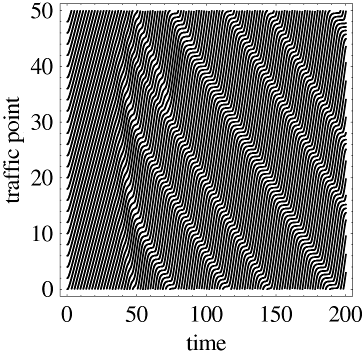
$$U(\Delta x) = \max(0, b(\Delta x - c) + a) - \max(0, b(\Delta x - c))$$



d-OV and u-OV as Traffic Models

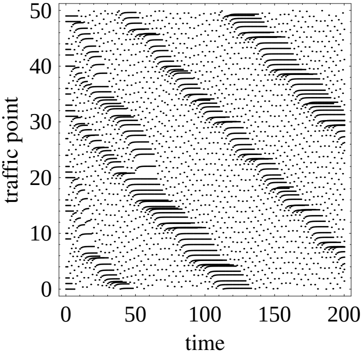
Discrete and ultradiscrete OV models work well as a traffic model.

d-OV



d-OV demo

u-OV



u-OV demo

Special Case of u-OV

A special case of u-OV is equivalent to the Fukui–Ishibashi model.

$$\boxed{x_k^{n+1} - 2x_k^n + x_k^{n-1} = A\{U(x_{k+1}^n - x_k^n) - \max(0, x_k^n - x_k^{n-1})}$$

$$\downarrow A = 1, \quad U(\Delta x) \geq 0, \quad \text{initial } x_k^n - x_k^{n-1} \geq 0$$

$$\boxed{x_k^{n+1} = x_k^n + U(x_{k+1}^n - x_k^n)} \qquad \text{(1st order)}$$

If $A = 1$, $a = v_{\mathsf{max}}$, $b = 1$ and $c = v_{\mathsf{max}} + 1$,

$$\boxed{U(\Delta x) = \max(0, \Delta x - 1) - \max(0, \Delta x - v_{\mathsf{max}} - 1)}$$

Then the above model is equivalent to the FI model.

New Relation Diagram

New relation diagram including both Euler and Lagrange expressions is obatained through the ultradiscretization.

Euler model		Lagrange model
		??? (PDE)
		$\uparrow$ continuous limit
Burgers (PDE)		OV (system of ODE's)
$\downarrow$ discretize		$\downarrow$ discretize
d-Burgers		d-OV
$\downarrow$ ultradiscretize		$\downarrow$ ultradiscretize
u-Burgers		u-OV
$\Cup$		$\Cup$
ECA184 (CA)	$\in$	Fukui–Ishibashi (CA)

What is a continuous limit of the OV model?
What is a relation of other known traffic models?
(NS model, slow start model, ...)