A New Expression of Soliton Solution to the Ultradiscrete Toda Equation

Hidetomo Nagai

Major in Pure and Applied Mathematics, Graduate School of Fundamental Science and Engineering, Waseda University, 3-4-1, Okubo, Shinjuku-ku, Tokyo 169-855, Japan

E-mail: `n1a9g8a1i@toki.waseda.jp`

Abstract. A new type of multi-soliton solution to the ultradiscrete Toda equation is proposed. The solution can be transformed into another expression of solution in a perturbation form. A direct proof of the solution is also given.

1. Introduction

Ultradiscretization is a limiting procedure to transform a discrete equation with continuous dependent variables into an ultradiscrete equation with discrete dependent variables[1]. Ultradiscrete system is applied to the traffic flow[2, 3] and the sorting algorithm[4] for example. An important formula of the procedure is a simple limiting formula,

$$\lim_{\epsilon \to +0} \epsilon \log(e^{a/\epsilon} + e^{b/\epsilon}) = \max(a, b). \tag{1}$$

For example, assume a discrete equation,

$$X_{n+1} = \frac{1 + X_n}{X_{n-1}}. \tag{2}$$

If we use a transformation of variable $X_n = e^{x_n/\epsilon}$ and take a limit $\epsilon \to +0$, we obtain an ultradiscrete equation,

$$x_{n+1} = \max(0, x_n) - x_{n-1}. \tag{3}$$

If initial values, for example, x_0 and x_1 are all integer, x_n for any n is also. The dependent variable X is considered to be discretized in this meaning.

If we apply this procedure to a multi-soliton solution to a discrete soliton equation, we obtain an ultradiscrete solution to an ultradiscrete soliton equation consistently. For example, a set of equation and solution in an ultradiscrete version is automatically derived by ultradiscretizing the discrete Korteweg–de Vries equation and its solution where the equation is transformed into a soliton cellular automaton called 'box and ball system' with a binary state value[1, 5].

However, there is a difficulty called 'negative problem' in the ultradiscretizing procedure. If '+' in the left side of (1) is replaced by '−', the limit by $\epsilon \to +0$ is not well-defined. Therefore, a multi-soliton solution expressed by a determinant can not be ultradiscretized though that in a perturbation form‡ can. A proof of discrete solution to discrete soliton equation is often realized by using an identity of determinants. Hence, it is important to find a 'determinant' form of ultradiscrete solution corresponding to determinant solutions. About this problem, Takahashi and Hirota showed a new expression of solution to the ultradiscrete KdV equation recently[6]. The solution is expressed by a form of ultradiscretized permanent which is defined by a signature-free determinant. It suggests that this type of solution is an ultradiscrete correspondence to a determinant one to a discrete soliton equation.

In this paper, the author proposes a new type of multi-soliton solution to the ultradiscrete Toda equation with a similar structure of expression. The discrete Toda equation introduced by Hirota[7] is

$$
\begin{aligned}
&\log(1 + V_{n+1}^m) - 2\log(1 + V_n^m) + \log(1 + V_{n-1}^m) \\
&= \log(1 + \delta^2 V_n^{m+1}) - 2\log(1 + \delta^2 V_n^m) + \log(1 + \delta^2 V_n^{m-1}).
\end{aligned}
\tag{5}
$$

Using transformations including parameters ϵ and L $(L > 0)$,

$$
V_n^m = e^{u_n^m/\epsilon}, \qquad \delta = e^{-L/2\epsilon},
\tag{6}
$$

and taking the limit $\epsilon \to +0$, we obtain the ultradiscrete Toda equation[8, 9]

$$
u_{n+1}^m - 2u_n^m + u_{n-1}^m = \max(0, u_n^{m+1} - L) - 2\max(0, u_n^m - L) + \max(0, u_n^{m-1} - L).
\tag{7}
$$

Its bilinear equation can also be ultradiscretized and given by

$$
f_{n+1}^m + f_{n-1}^m = \max(2f_n^m, f_n^{m+1} + f_n^{m-1} - L),
\tag{8}
$$

where u_n^m is related to f_n^m as

$$
u_n^m = f_n^{m+1} - 2f_n^m + f_n^{m-1}.
\tag{9}
$$

A multi-soliton solution in a perturbation form to (8) is reported in [9] and is derived by ultradiscretizing that to the discrete Toda equation. It follows a form,

$$
f_n^m = \max_{\mu_i=0,1}\left(\sum_{i=1}^N \mu_i s_i(m, n) + \sum_{1 \le i < j \le N} \mu_i \mu_j A_{ij}\right),
\tag{10}
$$

where $\max_{\mu_i=0,1} X(\mu_1, \mu_2, \ldots, \mu_N)$ denotes the maximum value of X in 2^N possible cases of $(\mu_1, \mu_2, \ldots, \mu_N)$ replacing each μ_i by 0 or 1. Detailed information on the right side is shown in Section 3. We refer to the above known type of solution as 'type I' in this paper.

‡ Hirota discovered multi-soliton solutions expressed by

$$
f = 1 + \varepsilon e^{\eta_1} + \varepsilon^2 e^{\eta_2} + \ldots + \varepsilon^n e^{\eta_n}
\tag{4}
$$

for arbitrary ε. In this paper, we call the above form and its ultradiscretized one 'perturbation form' referring Ref.[7].

We refer to the new type of solution proposed in this paper as 'type II'. The contents of this paper are as follows. In Section 2, a general form of solution of type II is proposed and a relation between both types is shown. In Section 3, a direct proof that the solution of type I satisfies (8) is given. Since solution of type II is equivalent to type I, this proof is also valid for type II. In Section 4, some conclusions are given.

2. New type of solution

A new type (type II) of multi-soliton solution to the ultradiscrete Toda equation is given by

$$f_n^m = \frac{1}{2}\overline{\max}(|s_1(m,n)|, |s_2(m,n)|, \ldots, |s_N(m,n)|), \tag{11}$$

where

$$s_i(m,n) = p_i m - \varepsilon_i q_i n + c_i, \tag{12}$$
$$q_i = \min(0, p_i + L) - \min(0, -p_i + L).$$

Parameters p_i and c_i are real and ε_i is 1 or -1. Function $\overline{\max}$ is defined by

$$\overline{\max}(\varphi_1(m,n), \varphi_2(m,n), \ldots, \varphi_N(m,n))$$
$$= \max_{\pi_i}(\varphi_{\pi_1}(m,n) + \varphi_{\pi_2}(m+1, n - \varepsilon_{\pi_1}) + \varphi_{\pi_3}(m+2, n - \varepsilon_{\pi_1} - \varepsilon_{\pi_2}) \tag{13}$$
$$+ \cdots + \varphi_{\pi_N}(m+N-1, n - \varepsilon_{\pi_1} - \varepsilon_{\pi_2} - \cdots - \varepsilon_{\pi_{N-1}}),$$

where $(\pi_1, \pi_2, \ldots, \pi_N)$ denotes an arbitrary permutation of natural numbers $1 \sim N$. For example, f_n^m in a case of $N = 3$ becomes

$$\begin{aligned}
f_n^m = \frac{1}{2}\max(\ &|s_1(m,n)| + |s_2(m+1, n - \varepsilon_1)| + |s_3(m+2, n - \varepsilon_1 - \varepsilon_2)|, \\
&|s_1(m,n)| + |s_2(m+2, n - \varepsilon_1 - \varepsilon_3)| + |s_3(m+1, n - \varepsilon_1)|, \\
&|s_1(m+1, n - \varepsilon_2)| + |s_2(m,n)| + |s_3(m+2, n - \varepsilon_1 - \varepsilon_2)|, \\
&|s_1(m+2, n - \varepsilon_2 - \varepsilon_3)| + |s_2(m,n)| + |s_3(m+1, n - \varepsilon_2)|, \\
&|s_1(m+1, n - \varepsilon_3)| + |s_2(m+2, n - \varepsilon_1 - \varepsilon_3)| + |s_3(m,n)|, \\
&|s_1(m+2, n - \varepsilon_2 - \varepsilon_3)| + |s_2(m+1, n - \varepsilon_3)| + |s_3(m,n)|).
\end{aligned} \tag{14}$$

Let us transform (11) to a solution of type I. Since the parameter q_i depends on p_i as

$$q_i = \begin{cases} -\min(0, -p_i + L) & (p_i \geq 0) \\ \min(0, p_i + L) & (p_i < 0) \end{cases}, \tag{15}$$

q_i is odd on p_i. Thus, (p_i, c_i) and $(-p_i, -c_i)$ give the same $|s_i(m,n)|$ and we may assume

$$p_1 \geq p_2 \geq \ldots \geq p_N > 0 \tag{16}$$

without loss of generality. Under this assumption, (11) reduces to

$$f_n^m = \frac{1}{2}\max_{\sigma_i = \pm 1}\left(\sum_{i=1}^{N}\sigma_i s_i + \sum_{i=1}^{N}\left(\frac{1+\sigma_i}{2}\left((N-i)p_i + q_i\varepsilon_i\sum_{j=i+1}^{N}\varepsilon_j\right)\right.\right.$$
$$\left.\left. + \sigma_i\sum_{j=1}^{i-1}\frac{1-\sigma_j}{2}(p_i + \varepsilon_i\varepsilon_j q_i)\right)\right), \tag{17}$$

where s_i denotes $s_i(m, n)$ for abbreviation. The proof of equivalence between (11) and (17) is shown in Appendix A. We obtain from (17),

$$f_n^m \leftrightarrow \frac{1}{2} \max_{\sigma_i = \pm 1} \left(\sum_{i=1}^{N} (1 + \sigma_i) s_i + \sum_{i=1}^{N} \left(\frac{1 + \sigma_i}{2} \left((N - i) p_i + q_i \varepsilon_i \sum_{j=i+1}^{N} \varepsilon_j \right) \right. \right.$$
$$\left. \left. + \sigma_i \sum_{j=1}^{i-1} \frac{1 - \sigma_j}{2} (p_i + \varepsilon_i \varepsilon_j q_i) \right) \right), \tag{18}$$

where $f_n^m \leftrightarrow g_n^m$ means that f_n^m and g_n^m give the same u_n^m through the right side of transformation (9). Moreover, (18) reduces to

$$f_n^m \leftrightarrow \max_{\mu_i = 0, 1} \left(\sum_{i=1}^{N} \mu_i \left(s_i + \frac{(N - i) p_i}{2} + q_i \varepsilon_i \sum_{j=i+1}^{N} \frac{1}{2} \varepsilon_j + \sum_{j=1}^{i-1} (p_i + \varepsilon_i \varepsilon_j q_i) \right. \right.$$
$$\left. \left. + \frac{1}{2} \sum_{j=i+1}^{N} (p_j + \varepsilon_i \varepsilon_j q_j) \right) - \sum_{1 \le i < j \le N} \mu_j \mu_i (p_j + \varepsilon_i \varepsilon_j q_j) \right) \tag{19}$$

using $\mu_i = (1 + \sigma_i)/2$. Hence, exchanging the free parameter c_i as

$$c_i + \frac{(N - i) p_i}{2} + q_i \varepsilon_i \sum_{j=i+1}^{N} \frac{1}{2} \varepsilon_j + \sum_{j=1}^{i-1} (p_i + \varepsilon_i \varepsilon_j q_i) + \frac{1}{2} \sum_{j=i+1}^{N} (p_j + \varepsilon_i \varepsilon_j q_j) \to c_i, \tag{20}$$

we obtain a solution of type I,

$$f_n^m \leftrightarrow \max_{\mu_i = 0, 1} \left(\sum_{i=1}^{N} \mu_i s_i - \sum_{1 \le i < j \le N} \mu_i \mu_j (p_j + \varepsilon_i \varepsilon_j q_j) \right). \tag{21}$$

Thus solutions of type I and II are equivalent.

3. Direct Proof of Solution of Type I

In this section, we prove that a solution of type I satisfies the ultradiscrete Toda equation. General form of the solution obeys

$$f_n^m = \max_{\mu_i = 0, 1} \left(\sum_{i=1}^{N} \mu_i s_i - \sum_{1 \le i < j \le N} \mu_i \mu_j A_{ij} \right), \tag{22}$$

where

$$s_i(m, n) = p_i m - \varepsilon_i q_i n + c_i, \qquad \varepsilon_i = \pm 1,$$
$$q_i = \min(0, p_i + L) - \min(0, -p_i + L), \tag{23}$$
$$A_{ij} = \min(\max(a_{ij}, -a_{ji}), \max(-a_{ij}, a_{ji})), \qquad a_{ij} = p_j + \varepsilon_i \varepsilon_j q_j.$$

In this form, p_i is arbitrary and f_n^m of (22) reduces to (21) under the condition (16).§ We give the following proposition about this solution.

Proposition 1 *Solution f_n^m defined by (22) satisfies a bilinear equation (8), that is,*

$$f_{n+1}^m + f_{n-1}^m = \max(2 f_n^m, f_n^{m+1} + f_n^{m-1} - L). \tag{24}$$

§ A mistake exists in the expression of A_{ij} shown in [9] and is corrected as shown in (23).

We can assume

$$|p_1| \geq |p_2| \geq \ldots \geq |p_N| > 0 \tag{25}$$

without loss of generality and then (22) is expressed by

$$f_n^m = \max_{\mu_i=0,1} \Big(\sum_{i=1}^N \mu_i s_i - \sum_{1 \leq i < j \leq N} \mu_i \mu_j \mathrm{sgn}(p_i p_j) |a_{ij}| \Big) \tag{26}$$

where

$$\mathrm{sgn}(x) = \begin{cases} 1 & (x > 0) \\ -1 & (x < 0) \end{cases}. \tag{27}$$

Substituting (26) into the left side of (24), we obtain

$$
\begin{aligned}
f_{n+1}^m + f_{n-1}^m = & \max_{\mu_i=0,1} \Big(\sum_{1 \leq i \leq N} \mu_i(s_i - \varepsilon_i q_i) - \sum_{1 \leq i < j \leq N} \mu_i \mu_j \mathrm{sgn}(p_i p_j) |a_{ij}| \Big) \\
& + \max_{\nu_i=0,1} \Big(\sum_{1 \leq i \leq N} \nu_i(s_i + \varepsilon_i q_i) - \sum_{1 \leq i < j \leq N} \nu_i \nu_j \mathrm{sgn}(p_i p_j) |a_{ij}| \Big).
\end{aligned}
\tag{28}
$$

Introducing new parameters λ_i and σ_i defined by

$$\lambda_i = \mu_i + \nu_i, \qquad \sigma_i = \mathrm{sgn}(p_i)(\mu_i - \nu_i), \tag{29}$$

for $1 \leq i \leq N$, we obtain

$$
\begin{aligned}
f_{n+1}^m + f_{n-1}^m = \max_{(\lambda_i,\sigma_i)} \Big(& \sum_{1 \leq i \leq N} \lambda_i s_i - \frac{1}{2} \sum_{1 \leq i < j \leq N} \lambda_i \lambda_j \mathrm{sgn}(p_i p_j) |a_{ij}| \\
& - \sum_{\substack{1 \leq i \leq N \\ \lambda_i=1}} \sigma_i \varepsilon_i \mathrm{sgn}(p_i) q_i - \frac{1}{2} \sum_{\substack{1 \leq i < j \leq N \\ \lambda_i=\lambda_j=1}} \sigma_i \sigma_j |a_{ij}| \Big).
\end{aligned}
\tag{30}
$$

Note that the pair (λ_i, σ_i) can be one of the following,

$$(0,0), \ (1,1), \ (1,-1), \ (2,0), \tag{31}$$

and $\max_{(\lambda_i,\sigma_i)} X(\lambda_1, \ldots, \lambda_N, \sigma_1, \ldots, \sigma_N)$ denotes the maximum value of X in 4^N possible cases of $(\lambda_1, \ldots, \lambda_N, \sigma_1, \ldots, \sigma_N)$ replacing each (λ_i, σ_i) by one of the above four pairs. Similarly, terms in the max function of the right side of (24) become respectively

$$2f_n^m = \max_{(\lambda_i,\sigma_i)} \Big(\sum_{1 \leq i \leq N} \lambda_i s_i - \frac{1}{2} \sum_{1 \leq i < j \leq N} \lambda_i \lambda_j \mathrm{sgn}(p_i p_j) |a_{ij}| - \frac{1}{2} \sum_{\substack{1 \leq i < j \leq N \\ \lambda_i=\lambda_j=1}} \sigma_i \sigma_j |a_{ij}| \Big), \tag{32}$$

$$
\begin{aligned}
f_n^{m+1} + f_n^{m-1} - L = \max_{(\lambda_i,\sigma_i)} \Big(& \sum_{1 \leq i \leq N} \lambda_i s_i - \frac{1}{2} \sum_{1 \leq i < j \leq N} \lambda_i \lambda_j \mathrm{sgn}(p_i p_j) |a_{ij}| \\
& + \sum_{\substack{1 \leq i \leq N \\ \lambda_i=1}} \sigma_i \mathrm{sgn}(p_i) p_i - \frac{1}{2} \sum_{\substack{1 \leq i < j \leq N \\ \lambda_i=\lambda_j=1}} \sigma_i \sigma_j |a_{ij}| \Big) - L.
\end{aligned}
\tag{33}
$$

Comparing terms in the max functions of (30), (32) and (33), (24) holds if

$$\max_{\sigma_i=\pm1}\left(-\sum_{1\le i\le N}\sigma_i\varepsilon_i\mathrm{sgn}(p_i)q_i-\frac{1}{2}\sum_{1\le i<j\le N}\sigma_i\sigma_j|a_{ij}|\right)$$
$$=\max\left(\max_{\sigma_i=\pm1}\left(-\frac{1}{2}\sum_{1\le i<j\le N}\sigma_i\sigma_j|a_{ij}|\right),\right.\tag{34}$$
$$\left.\max_{\sigma_i=\pm1}\left(\sum_{1\le i\le N}\sigma_i\mathrm{sgn}(p_i)p_i-\frac{1}{2}\sum_{1\le i<j\le N}\sigma_i\sigma_j|a_{ij}|\right)-L\right).$$

Considering a symmetry of expression about p_i and q_i, we can assume $p_1\ge p_2\ge\dots\ge p_N>0$. Under this assumption, q_i is given by

$$q_i=-\min(0,-p_i+L)\tag{35}$$

and $0\le q_i\le p_i$. Moreover, $a_{ij}=p_j+\varepsilon_i\varepsilon_jq_j\ge0$. Hence, we give the following proposition to prove.

Proposition 2 *Functions* $g_l(\sigma_i)$, $g_{r_1}(\sigma_i)$ *and* $g_{r_2}(\sigma_i)$ *defined by*

$$g_l(\sigma_i)\ =-\sum_{i=1}^N\sigma_i\varepsilon_iq_i-\frac{1}{2}\sum_{1\le i<j\le N}\sigma_i\sigma_ja_{ij},$$
$$g_{r_1}(\sigma_i)=-\frac{1}{2}\sum_{1\le i<j\le N}\sigma_i\sigma_ja_{ij},\tag{36}$$
$$g_{r_2}(\sigma_i)=\sum_{i=1}^N\sigma_ip_i-\frac{1}{2}\sum_{1\le i<j\le N}\sigma_i\sigma_ja_{ij},$$

satisfy

$$\max_{\sigma_i=\pm1}g_l(\sigma_i)=\max(\max_{\sigma_i=\pm1}g_{r_1}(\sigma_i),\ \max_{\sigma_i=\pm1}g_{r_2}(\sigma_i)-L),\tag{37}$$

for any N *and* p_i $(p_1\ge p_2\ge\dots\ge p_N>0)$.

Define maximum values of $g_l(\sigma_i)$, $g_{r_1}(\sigma_i)$ and $g_{r_2}(\sigma_i)$ by $\bar{g}_l$, $\bar{g}_{r_1}$ and $\bar{g}_{r_2}$ respectively. They are given if σ_i satisfies the following conditions,

$$\bar{g}_l\ :\quad\sigma_i=(-1)^i\prod_{l=1}^i\varepsilon_l\quad(i:\text{odd}),\qquad(-1)^i\prod_{l=1}^{i-1}\varepsilon_l\quad(i:\text{even}),$$

$$\bar{g}_{r_1}:\quad\sigma_i=\prod_{l=1}^i\varepsilon_l\quad(i:\text{odd}),\qquad-\prod_{l=1}^{i-1}\varepsilon_l\quad(i:\text{even}),$$

$$\bar{g}_{r_2}:\quad\sigma_i=1\quad(i=1),\qquad(-1)^i\varepsilon_1\prod_{l=1}^i\varepsilon_l\quad(i:\text{odd},\ i\ge3),\tag{38}$$

$$(-1)^i\varepsilon_1\prod_{l=1}^{i-1}\varepsilon_l\quad(i:\text{even}).$$

Substituting these conditions, we can derive

$$\bar{g}_l = \sum_{i=1}^{N}\frac{1+(-1)^i}{4}p_i + q_1 - \sum_{i=2}^{N}\Big(\frac{1+(-1)^i}{4}\prod_{l=1}^{i}\varepsilon_l - \frac{1+(-1)^{i-1}}{4}\Big(1+\prod_{l=1}^{i-1}\varepsilon_l\Big)\Big)q_i, \quad (39)$$

$$\bar{g}_{r_1} = \sum_{i=1}^{N}\frac{1+(-1)^i}{4}p_i + \sum_{i=2}^{N}\Big(\frac{1+(-1)^i}{4}\prod_{l=1}^{i}\varepsilon_l + \frac{1+(-1)^{i-1}}{4}\Big(1-\prod_{l=1}^{i-1}\varepsilon_l\Big)\Big)q_i, \quad (40)$$

$$\bar{g}_{r_2} = p_1 + \sum_{i=1}^{N}\frac{1+(-1)^i}{4}p_i - \sum_{i=2}^{N}\Big(\frac{1+(-1)^i}{4}\prod_{l=1}^{i}\varepsilon_l - \frac{1+(-1)^{i-1}}{4}\Big(1+\prod_{l=1}^{i-1}\varepsilon_l\Big)\Big)q_i. \quad (41)$$

Though the whole proof on the above relations (39)~(41) is long and omitted, an important formula for the proof is shown in Appendix B.

If $p_1 \geq L$, then we obtain

$$\bar{g}_{r_1} - \bar{g}_l = -q_1 + \sum_{i=2}^{N}\Big(\frac{1+(-1)^i}{2}\prod_{l=1}^{i}\varepsilon_l - \frac{1+(-1)^{i-1}}{2}\prod_{l=1}^{i-1}\varepsilon_l\Big)q_i$$
$$\leq -q_1 + q_2 - q_3 + q_4 - \ldots + (-1)^N q_N \leq 0, \quad (42)$$

and

$$\bar{g}_{r_2} - \bar{g}_l - L = p_1 - q_1 - L = 0. \quad (43)$$

If $p_1 < L$, then $q_i = 0$ for any $1 \leq i \leq N$ and

$$\bar{g}_{r_1} - \bar{g}_l = 0, \qquad \bar{g}_{r_2} - \bar{g}_l - L = p_1 - L \leq 0. \quad (44)$$

Thus a relation

$$\max(\bar{g}_{r_1} - \bar{g}_l, \ \bar{g}_{r_2} - \bar{g}_l - L) = 0, \quad (45)$$

holds and (37) is satisfied.

4. Conclusion

A new expression of multi-soliton solution to the ultradiscrete Toda equation is proposed. The solution is equivalent to a solution in a perturbation form. A direct proof of solution is also given.

The former solution expressed by $\overline{\max}$ in (11) can be derived by ultradiscretization. For example, let us consider

$$F = \sum_{\pi_i}\psi_{\pi_1}(m,n)\psi_{\pi_2}(m+1,n-\varepsilon_{\pi_1})\psi_{\pi_3}(m+2,n-\varepsilon_{\pi_1}-\varepsilon_{\pi_2})$$
$$\cdots\psi_{\pi_N}(m+N-1,n-\varepsilon_{\pi_1}-\varepsilon_{\pi_2}-\cdots-\varepsilon_{\pi_{N-1}}). \quad (46)$$

If we use a transformation

$$\psi_i(m,n) = \cosh(s_i(m,n)/2\varepsilon), \quad (47)$$

then we give the right side of (11) by

$$\lim_{\epsilon\to+0}\epsilon\log F. \quad (48)$$

It suggests that a discrete solution as a correspondence of ultradiscrete solution can exists. However, such a solution has not yet been found. Moreover, proof of discrete solution to discrete soliton equation is often realized by using an identity of determinants. To find the discrete correspondence of solution and to prove an ultradiscrete solution by some ultradiscretized identity are considered to be important future problems.

Acknowledgments

Appendix A. Equivalence between (11) and (17)

In this appendix, we prove the following proposition.

Proposition 3

$$\overline{\max}(|s_1(m, n)|, |s_2(m, n)|, \ldots, |s_N(m, n)|)$$

$$= \max_{\sigma_i = \pm 1} \left(\sum_{i=1}^{N} \sigma_i s_i + \sum_{i=1}^{N} \left(\frac{1 + \sigma_i}{2} \left((N - i)p_i + q_i \varepsilon_i \sum_{j=i+1}^{N} \varepsilon_j \right) \right. \right.$$

$$\left. \left. + \sigma_i \sum_{j=1}^{i-1} \frac{1 - \sigma_j}{2} (p_i + \varepsilon_i \varepsilon_j q_i) \right) \right) \tag{A.1}$$

where

$$s_i(m, n) = p_i m - \varepsilon_i q_i n + c_i, \qquad -1 \leq \varepsilon_i \leq 1, \tag{A.2}$$

$$p_i \geq 0, \qquad q_i \geq 0, \qquad p_1 - q_1 \geq p_2 - q_2 \geq \ldots \geq p_N - q_N \geq 0. \tag{A.3}$$

If p_i and q_i satisfy (15) and (16), conditions of (A.3) is all satisfied. Therefore, if the above proposition is proved, equivalence between (11) and (17) is shown. Note that the key formula shown in [6] is a special case of (A.1) with $\varepsilon_i = \varepsilon_j$ for any $1 \leq i, j \leq N$.

Proof. The left side of (A.1) is equal to

$$\max_{\pi_i} \sum_{i=1}^{N} \left| s_{\pi_i} + (i - 1)p_{\pi_i} + q_{\pi_i} \varepsilon_{\pi_i} \sum_{j=1}^{i-1} \varepsilon_{\pi_j} \right|, \tag{A.4}$$

where $(\pi_1, \pi_2, \ldots, \pi_N)$ is a permutation of natural numbers $1 \sim N$. Noting $|x| = \max(x, -x)$, the above expression is rewritten by

$$\max_{\pi_i} \left(\max_{\sigma_{\pi_i} = \pm 1} \sum_{i=1}^{N} \sigma_{\pi_i} \left(s_{\pi_i} + (i - 1)p_{\pi_i} + q_{\pi_i} \varepsilon_{\pi_i} \sum_{j=1}^{i-1} \varepsilon_{\pi_j} \right) \right)$$

$$= \max_{\sigma_i = \pm 1} \left(\sum_{i=1}^{N} \sigma_i s_i + \max_{\pi_i} \sum_{i=1}^{N} \sigma_{\pi_i} \left((i - 1)p_{\pi_i} + q_{\pi_i} \varepsilon_{\pi_i} \sum_{j=1}^{i-1} \varepsilon_{\pi_j} \right) \right). \tag{A.5}$$

Thus, what to prove is the following.

$$\max_{\pi_i} \sum_{i=1}^{N} \sigma_{\pi_i}\left((i-1)p_{\pi_i} + q_{\pi_i}\varepsilon_{\pi_i}\sum_{j=1}^{i-1}\varepsilon_{\pi_j}\right)$$
$$= \sum_{i=1}^{N}\left(\frac{1+\sigma_i}{2}\left((N-i)p_i + q_i\varepsilon_i\sum_{j=i+1}^{N}\varepsilon_j\right) + \sigma_i\sum_{j=1}^{i-1}\frac{1-\sigma_j}{2}(p_i + \varepsilon_i\varepsilon_j q_i)\right). \tag{A.6}$$

We use a mathematical induction on N. The claim is trivial for $N = 1$. Then, let us assume (A.6) holds for a certain N and define $L(\pi_1, \pi_2, \ldots, \pi_{N+1})$ by

$$L(\pi_1, \pi_2, \ldots, \pi_{N+1}) = \sum_{i=1}^{N+1} \sigma_{\pi_i}\left((i-1)p_{\pi_i} + q_{\pi_i}\varepsilon_{\pi_i}\sum_{j=1}^{i-1}\varepsilon_{\pi_j}\right). \tag{A.7}$$

For $N + 1$, assume a certain permutation $(\pi_1, \pi_2, \ldots, \pi_{N+1})$ and let α be a number satisfying $1 \le \alpha \le N + 1$ and $\pi_\alpha = 1$. In the case of $\sigma_1 = 1$, we have

$$L(\pi_1, \ldots, \pi_{\alpha-1}, \pi_{\alpha+1}, \ldots, \pi_{N+1}, \pi_\alpha) - L(\pi_1, \pi_2, \ldots, \pi_{N+1})$$
$$= (N - \alpha + 1)p_1 + q_1\varepsilon_1 \sum_{j=\alpha+1}^{N+1} \varepsilon_{\pi_j} - \sum_{i=\alpha+1}^{N+1} \sigma_{\pi_i}(p_{\pi_i} + \varepsilon_1\varepsilon_{\pi_i}q_{\pi_i})$$
$$= \sum_{i=\alpha+1}^{N+1} (p_1 - \sigma_{\pi_i}p_{\pi_i} + \varepsilon_1\varepsilon_{\pi_i}(q_1 - \sigma_{\pi_i}q_{\pi_i}))$$
$$\ge 0. \tag{A.8}$$

In the case of $\sigma_1 = -1$, we have

$$L(\pi_\alpha, \pi_1, \ldots, \pi_{\alpha-1}, \pi_{\alpha+1}, \ldots, \pi_{N+1}) - L(\pi_1, \pi_2, \ldots, \pi_{N+1})$$
$$= \sum_{i=1}^{\alpha-1}(p_1 + \varepsilon_1\varepsilon_{\pi_i}q_1 + \sigma_{\pi_i}(p_{\pi_i} + \varepsilon_1\varepsilon_{\pi_i}q_{\pi_i})) \tag{A.9}$$
$$\ge 0.$$

Thus, we obtain

$$\max_{\pi_i} L(\pi_1, \pi_2, \ldots, \pi_{N+1})$$

$$= \frac{1+\sigma_1}{2} \max_{\pi_i} L(\pi_1 + 1, \ldots, \pi_N + 1, 1) + \frac{1-\sigma_1}{2} \max_{\pi_i} L(1, \pi_1 + 1, \ldots, \pi_N + 1)$$

$$= \frac{1+\sigma_1}{2} \left(N p_1 + q_1 \varepsilon_1 \sum_{j=2}^{N+1} \varepsilon_j \right.$$

$$\left. + \max_{\pi_i} \sum_{i=1}^{N} \sigma_{\pi_i+1} \left((i-1) p_{\pi_i+1} + q_{\pi_i+1} \varepsilon_{\pi_i+1} \sum_{j=1}^{i-1} \varepsilon_{\pi_j+1} \right) \right)$$

$$+ \frac{1-\sigma_1}{2} \left(\max_{\pi_i} \sum_{i=1}^{N} \sigma_{\pi_i+1} \left((i-1) p_{\pi_i+1} + q_{\pi_i+1} \varepsilon_{\pi_i+1} \sum_{j=1}^{i-1} \varepsilon_{\pi_j+1} \right) \right. \tag{A.10}$$

$$\left. + \sum_{i=1}^{N} \sigma_{\pi_i+1} (p_{\pi_i+1} + \varepsilon_1 \varepsilon_{\pi_i+1} q_{\pi_i+1}) \right)$$

$$= \max_{\pi_i} \sum_{i=1}^{N} \sigma_{\pi_i+1} \left((i-1) p_{\pi_i+1} + q_{\pi_i+1} \varepsilon_{\pi_i+1} \sum_{j=1}^{i-1} \varepsilon_{\pi_j+1} \right)$$

$$+ \frac{1+\sigma_1}{2} \left(N p_1 + q_1 \varepsilon_1 \sum_{j=2}^{N+1} \varepsilon_j \right) + \frac{1-\sigma_1}{2} \sum_{i=1}^{N} \sigma_{\pi_i+1} (p_{\pi_i+1} + \varepsilon_1 \varepsilon_{\pi_i+1} q_{\pi_i+1}).$$

Using the assumption of the induction, (A.10) reduces to

$$\sum_{i=2}^{N+1} \left(\frac{1+\sigma_i}{2} \left((N+1-i) p_i + q_i \varepsilon_i \sum_{j=i+1}^{N} \varepsilon_j \right) + \sigma_i \sum_{j=2}^{i-1} \frac{1-\sigma_j}{2} \left(p_i + \varepsilon_i \varepsilon_j q_i \right) \right)$$

$$+ \frac{1+\sigma_1}{2} \left(N p_1 + q_1 \varepsilon_1 \sum_{j=2}^{N+1} \varepsilon_j \right) + \frac{1-\sigma_1}{2} \left(\sum_{i=2}^{N+1} \sigma_i (p_i + \varepsilon_1 \varepsilon_i q_i) \right) \tag{A.11}$$

$$= \sum_{i=1}^{N+1} \left(\frac{1+\sigma_i}{2} \left((N-i) p_i + q_i \varepsilon_i \sum_{j=i+1}^{N} \varepsilon_j \right) + \sigma_i \sum_{j=1}^{i-1} \frac{1-\sigma_j}{2} \left(p_i + \varepsilon_i \varepsilon_j q_i \right) \right).$$

This is equal to the right side of (A.6) for $N + 1$ and the induction holds. Thus this completes the proof of Prop.3. □

Appendix B. Proposition for proof on (39)∼(41)

To prove (39)∼(41), the following proposition is important. The proof on the proposition is shown in this appendix.

Proposition 4
Define $g(\sigma_1, \sigma_2, \ldots, \sigma_N)$ by

$$g(\sigma_1, \sigma_2, \ldots, \sigma_N) = - \sum_{1 \le i < j \le N} \sigma_i \sigma_j p_j, \tag{B.1}$$

where N is a natural number, σ_i is $+1$ or -1 and

$$p_1 \geq p_2 \geq \ldots \geq p_N > 0. \tag{B.2}$$

Then,

$$g(\sigma_1, \sigma_2, \ldots, \sigma_{N-n-1}, \bar{\sigma}_{N-n}, \bar{\sigma}_{N-n+1}, \ldots, \bar{\sigma}_N) \geq g(\sigma_1, \sigma_2, \ldots, \sigma_N) \tag{B.3}$$

for any $0 \leq n \leq N - 2$, where $\bar{\sigma}_i$ is given by

$$\bar{\sigma}_i = \begin{cases} 1 & (\sum_{l=1}^{N-n-1} \sigma_l + \sum_{l=N-n}^{i-1} \bar{\sigma}_l \leq 0) \\ -1 & (\sum_{l=1}^{N-n-1} \sigma_l + \sum_{l=N-n}^{i-1} \bar{\sigma}_l > 0) \end{cases} \tag{B.4}$$

Proof. We use a mathematical induction on n. For $n = 0$, we have

$$g(\sigma_1, \ldots, \sigma_{N-1}, \bar{\sigma}_N) - g(\sigma_1, \ldots, \sigma_{N-1}, -\bar{\sigma}_N) = -2\bar{\sigma}_N p_N \sum_{i=1}^{N-1} \sigma_i \geq 0, \tag{B.5}$$

and (B.3) holds. Assume (B.3) for a certain $n = k$ $(0 \leq k \leq N - 2)$, that is,

$$g(\sigma_1, \ldots, \sigma_{N-k-1}, \bar{\sigma}_{N-k}, \bar{\sigma}_{N-k+1}, \ldots, \bar{\sigma}_N) \geq g(\sigma_1, \ldots, \sigma_N). \tag{B.6}$$

Then, what to prove for induction in the case of $n = k + 1$ is

$$\begin{aligned} g(\sigma_1, \ldots, \sigma_{N-k-2}, \bar{\sigma}_{N-k-1}, \bar{\sigma}_{N-k}, \ldots, \bar{\sigma}_N) \\ -g(\sigma_1, \ldots, \sigma_{N-k-2}, -\bar{\sigma}_{N-k-1}, \hat{\sigma}_{N-k}, \ldots, \hat{\sigma}_N) \geq 0, \end{aligned} \tag{B.7}$$

where

$$\begin{aligned} \bar{\sigma}_{N-k-1} &= \begin{cases} 1 & (\sum_{l=1}^{N-k-2} \sigma_l \leq 0) \\ -1 & (\sum_{l=1}^{N-k-2} \sigma_l > 0) \end{cases}, \\ \bar{\sigma}_i &= \begin{cases} 1 & (\sum_{l=1}^{N-k-2} \sigma_l + \bar{\sigma}_{N-k-1} + \sum_{l=N-k}^{i-1} \bar{\sigma}_l \leq 0) \\ -1 & (\sum_{l=1}^{N-k-2} \sigma_l + \bar{\sigma}_{N-k-1} + \sum_{l=N-k}^{i-1} \bar{\sigma}_l > 0) \end{cases}, \\ \hat{\sigma}_i &= \begin{cases} 1 & (\sum_{l=1}^{N-k-2} \sigma_l - \bar{\sigma}_{N-k-1} + \sum_{l=N-k}^{i-1} \hat{\sigma}_l \leq 0) \\ -1 & (\sum_{l=1}^{N-k-2} \sigma_l - \bar{\sigma}_{N-k-1} + \sum_{l=N-k}^{i-1} \hat{\sigma}_l > 0) \end{cases}. \end{aligned} \tag{B.8}$$

The left side of (B.7) reduces to

$$\begin{aligned} -\Big(2 \sum_{i=1}^{N-k-2} \sigma_i \bar{\sigma}_{N-k-1} p_{N-k-1} + \sum_{i=1}^{N-k-2} \sum_{j=N-k}^{N} \sigma_i(\bar{\sigma}_j - \hat{\sigma}_j) p_j \\ + \bar{\sigma}_{N-k-1} \sum_{j=N-k}^{N} (\bar{\sigma}_j + \hat{\sigma}_j) p_j + \sum_{i=N-k}^{N} \sum_{j=i+1}^{N} (\bar{\sigma}_i \bar{\sigma}_j - \hat{\sigma}_i \hat{\sigma}_j) p_j \Big). \end{aligned} \tag{B.9}$$

Let us define S by $\sum_{i=1}^{N-k-2} \sigma_i$ and consider two cases of $S > 0$ or $S \leq 0$.

In the case of $S > 0$, $\bar{\sigma}_i = -1$ for $N - k - 1 \leq i \leq N - k + S - 1$ since $S + \sum_{l=N-k-1}^{i-1} \bar{\sigma}_l > 0$, and $\hat{\sigma}_i = -1$ for $N - k \leq i \leq N - k + S + 1$. Therefore $\bar{\sigma}_i$ and $\hat{\sigma}_i$ are given by

$$\begin{aligned} (\bar{\sigma}_{N-k-1}, \bar{\sigma}_{N-k}, \ldots, \bar{\sigma}_N) &= (\underbrace{-1, -1, \ldots, -1}_{S}, 1, -1, 1, -1, 1, \ldots), \\ (-\bar{\sigma}_{N-k-1}, \hat{\sigma}_{N-k}, \ldots, \hat{\sigma}_N) &= (1, \underbrace{-1, -1, \ldots, -1}_{S}, -1, 1, -1, 1, \ldots). \end{aligned} \tag{B.10}$$

Hence,

$$\hat{\sigma}_i = \begin{cases} -\bar{\sigma}_i & (i = N - k + S - 1) \\ \bar{\sigma}_i & (\text{otherwise}) \end{cases}, \tag{B.11}$$

and (B.9) reduces to

$$
-\left(-2Sp_{N-k-1} + 2Sp_{N-k+S-1} - 2\sum_{j=N-k}^{N-k+S-2}\bar{\sigma}_j p_j - 2\sum_{j=N-k+S}^{N}\bar{\sigma}_j p_j \right.
$$
$$
\left. + 2\sum_{j=N-k+S}^{N}\bar{\sigma}_j p_j + 2p_{N-k+S-1}\sum_{i=N-k}^{N-k+S-2}\bar{\sigma}_i \right)
\tag{B.12}
$$
$$
=2\left(Sp_{N-k-1} - Sp_{N-k+S-1} - \sum_{j=N-k}^{N-k+S-2}p_j + (S-1)p_{N-k+S-1}\right)
$$
$$
\geq 2\left(Sp_{N-k-1} - Sp_{N-k+S-1} - (S-1)p_{N-k} + (S-1)p_{N-k+S-1}\right)
$$
$$
=2(p_{N-k} - p_{N-k+S-1}) \geq 0.
$$

Similarly, in the case of $S \leq 0$, $\bar{\sigma}_i$ and $\hat{\sigma}_i$ are given by

$$
\begin{aligned}
(\bar{\sigma}_{N-k-1}, \bar{\sigma}_{N-k}, \ldots, \bar{\sigma}_N) &= (\underbrace{1, 1, \ldots, 1}_{-S}, 1, -1, 1, -1, 1, \ldots), \\
(-\bar{\sigma}_{N-k-1}, \hat{\sigma}_{N-k}, \ldots, \hat{\sigma}_N) &= (-1, \underbrace{1, 1, \ldots, 1}_{-S}, 1, 1, -1, 1, \ldots).
\end{aligned}
\tag{B.13}
$$

Hence,

$$\hat{\sigma}_i = \begin{cases} -\bar{\sigma}_i & (i = N - k - S) \\ \bar{\sigma}_i & (\text{otherwise}) \end{cases}, \tag{B.14}$$

and (B.9) reduces to

$$
-\left(2Sp_{N-k-1} - 2Sp_{N-k-S} + 2\sum_{j=N-k}^{N-k-S-1}\bar{\sigma}_j p_j + 2\sum_{j=N-k-S+1}^{N}\bar{\sigma}_j p_j \right.
$$
$$
\left. - 2\sum_{j=N-k-S+1}^{N}\bar{\sigma}_j p_j - 2\sum_{i=N-k}^{N-k-S-1}\bar{\sigma}_i p_{N-k-S}\right)
\tag{B.15}
$$
$$
=2\left(-Sp_{N-k-1} + Sp_{N-k-S} - \sum_{j=N-k}^{N-k-S-1}p_j - Sp_{N-k-S}\right)
$$
$$
\geq 2S\left(-p_{N-k-1} + p_{N-k-S} + p_{N-k} - p_{N-k-S}\right) \geq 0.
$$

As a result, (B.7) holds and this completes the proof of Prop.4. $\square$

References

[1] T Tokihiro, D Takahashi, J Matsukidaira and J Satsuma, *Phys. Rev. Lett.* **76** (1996) 3247–3250.

[2] K Nishinari, D Takahashi, *J. Phys. A: Math. Gen.* **31** (1998) 5439–5450

[3] K Nishinari, D Takahashi, *J. Phys. A: Math. Gen.* **32** (1999) 93–104

[4] A Nagai, D Takahashi and T Tokihiro, *Phys. Lett. A* **255** (1999) 265–271.

[5] D Takahashi and J Satsuma, *J. Phys. Soc. Japan* **59** (1990) 3514–3519.

[6] D Takahashi, R Hirota, *J. Phys. Soc. Japan* **76** (2007) 104007–104012

[7] R Hirota, *J. Phys. Soc. Japan* **43** (1977) 2074–2078.

[8] D Takahashi and J Matsukidaira, *Phys. Lett. A* **209** (1995) 184–188.

[9] J Matsukidaira, J Satsuma, D Takahashi, T Tokihiro, and M Torii, *Phys. Lett. A* **225** (1997) 287–295.