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ソリトン方程式の不安定解の超離散極限

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ソリトン方程式の不安定解の
超離散極限

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Abstract

差分 KdV 方程式や戸田方程式の解 u は 2 種類に分類される。一つは値が $u > 1$ となる解で超離散化すると $U > 0$ である。これを正の解と呼ぶ。もう一つは値が $1 > u > 0$ となる解で超離散化すると $U < 0$ である。これを負の解と呼ぶ。差分方程式の負の解は不安定な解である。しかし負の解は超離散化すると安定な解になる。正の解と負の解の衝突によって新しい現象が観測される。

1 Introduction

We have a discrete KdV equation

$$\frac{1}{u_{n+1}^{m+1}} - \frac{1}{u_n^m} = \delta(u_{n+1}^m - u_n^{m+1}), \tag{1}$$

which is known to be reduced, in the ultradiscrete limit, to the well-known Box and Ball system (Takahashi and Hirota (2007)),

$$U_n^{m+1} = \min\left(1 - U_n^m, \sum_{k=-\infty}^{n-1} (U_k^m - Um + 1_k)\right).$$

We call Eq.(1) BBB equation (Back ultradiscretization of the Box and Ball system). Let

$$u_n^m = \frac{f_{n+1}^m f_n^{m+1}}{f_n^m f_{n+1}^{m+1}}.$$

Then BBB equation is transformed into the bilinear form

$$f_n^{m-1} f_{n+1}^{m+1} = \delta f_{n+1}^{m-1} f_n^{m+1} + (1-\delta) f_n^m f_{n+1}^m.$$

In order to consider BBB equation under the periodic boundary condition we introduce a new dependent variable z_n^m defined by

$$z_n^m = \frac{1-\delta}{1-\delta u_n^{m-1} u_n^m} = \frac{f_n^{m-1} f_{n+1}^{m+1}}{f_n^m f_{n+1}^m}. \tag{2}$$

We find

$$\log z_n^m = (e^{\partial n} - e^{-\partial m})(e^{\partial m} - 1) \log f_n^m.$$

On the other hand we have

$$\log u_n^m = (1 - e^{\partial n})(e^{\partial m} - 1) \log f_n^m.$$

Accordingly we find

$$\frac{u_{n+1}^m}{u_n^{m-1}} = \frac{z_n^m}{z_{n+1}^m}. \tag{3}$$

Using Eqs.(2) and (3),we obtain

$$z_n^{m+1} = 1 - \delta + \delta u_n^m u_{n-1}^m z_{n-1}^{m+1}, \tag{4}$$

$$u_{n+1}^{m+1} z_{n+1}^{m+1} = u_n^m z_n^{m+1}. \tag{5}$$

We consider BBB equation under the periodic boundary condition,

$$u_{n+N}^m = u_n^m, \quad z_{n+N}^m = z_n^m.$$

Let $a_n = \delta u_n^m u_{n-1}^m$. Then Eq.(4) reads

$$z_n^{m+1} - a_n z_{n-1}^{m+1} = 1, \text{ for } n = 0, 1, 2, \cdots, N-1.$$

Solutions to the linear equation are given by

$$z_n^{m+1} = \frac{1}{\Delta}\left[1 + \sum_{k=1}^{N-1}\left(\prod_{s=0}^{k-1} a_{n-s}\right)\right], \qquad \text{for } n = 0, 1, 2, \cdots, N-1,$$

where $\Delta = 1 - \prod_{s=0}^{N-1} a_s$.
Substituting z_n^{m+1} into Eq.(5),we obtain an expression for u_n^{m+1},

$$u_n^{m+1} = u_{n-1}^m \frac{\left[1 + \sum_{k=1}^{N-1}\left(\prod_{s=0}^{k-1} a_{n-1-s}^m\right)\right]}{\left[1 + \sum_{k=1}^{N-1}\left(\prod_{s=0}^{k-1} a_{n-s}^m\right)\right]}, \tag{6}$$

($a_n^m = \delta u_n^m u_{n-1}^m$), which gives an explicit mapping of u_n^m under periodic boundary conditions.
Equation (6) is easily ultradiscretized,

$$\begin{aligned} U_n^{m+1} &= U_{n-1}^m \\ &+ \max_{k=0}^{N-1}\left(\sum_{s=0}^{k-1} A_{n-1-s}^m\right) - \max_{k=0}^{N-1}\left(\sum_{s=0}^{k-1} A_{n-s}^m\right), \end{aligned}$$

where $A_n^m = U_n^m + U_{n-1}^m - 1$.

We remark that ultradiscrete BBB system is equal to the Box and Ball system. However we consider that the number of balls in a box in ultradiscrete BBB system is arbitrary. It could be over the capacity of the box or a nagative number.
What happens if we put *three* balls in a box of the capacity 1 ?

```
{0,0,0,0,0,0,3,0,0,0,0,0,0,0}

{0,0,0,0,0,0,-2,1,1,1,1,1,0,0}

{1,1,1,0,0,0,0,-2,0,0,0,0,1,1}

{0,0,0,1,1,1,1,1,-2,0,0,0,0,0}

{0,0,0,0,0,0,0,0,3,0,0,0,0,0}

{0,0,0,0,0,0,0,0,-2,1,1,1,1,1}

{1,1,1,1,1,0,0,0,0,-2,0,0,0,0}

{0,0,0,0,0,1,1,1,1,1,-2,0,0,0}

{0,0,0,0,0,0,0,0,0,0,3,0,0,0}

{1,1,0,0,0,0,0,0,0,0,-2,1,1,1}

{0,0,1,1,1,1,1,0,0,0,0,-2,0,0}

{0,0,0,0,0,0,0,1,1,1,1,1,-2,0}
```

```
{0,0,0,0,0,0,0,0,0,0,0,0,3,0}

{1,1,1,1,0,0,0,0,0,0,0,0,-2,1}

{0,0,0,0,1,1,1,1,1,0,0,0,0,-2}

{-2,0,0,0,0,0,0,0,0,1,1,1,1,1}

{3,0,0,0,0,0,0,0,0,0,0,0,0,0}

{-2,1,1,1,1,1,0,0,0,0,0,0,0,0}
```

We observe two types of solitons. One is a usual soliton (of 5 balls) moving with speed 5 and another is a negative-soliton of magnitude (-2) moving with speed 1.

What are the negative-solitons ?
Are their any solutions to the BBB equation which are reduced to the negative-solitons in the ultradiscrete limit ?
We know τ-functions describing soliton-soliton interactions.
What are τ-functions, in the ultradiscrete limit, describing collision of solitons(balls) with negative-solitons ?

2 Review of the BBB equation

We show in the followings that solitons of the BBB equation are reduced to balls in the Box and Ball system in the ultradiscrete limit ($\epsilon \to 0$).
One-soliton solution to the bilinear equation is given by

$$\begin{aligned} &f_n^m = 1 + a_1 e^{\eta_1}, \quad e^{\eta_1} = \omega_1^m p_1^n. \\ &p_1 = (1 + \delta\omega_1)/(\omega_1 + \delta), \end{aligned}$$

Let U_n^m,F_n^m and Ω_1 be ultradiscrete limit of u_n^m, f_n^m and ω_1,

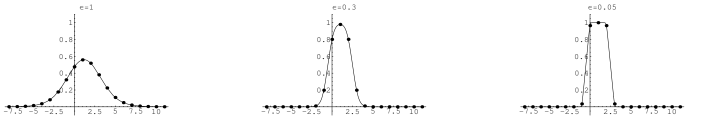
$$\begin{aligned} &U_n^m = \epsilon \log u_n^m, \quad F_n^m = \epsilon \log f_n^m, \\ &\Omega_1 = \epsilon \log \omega_1, \quad \delta = e^{-1/\epsilon} \end{aligned}$$

The next figures show how the one-soliton solution is reduced to the balls in the Box and Ball system as $\epsilon \to 0$ for $\Omega_1 = 3$ and 5.

Graphs of $U_n^m(\epsilon) = \epsilon \log u_n^m$

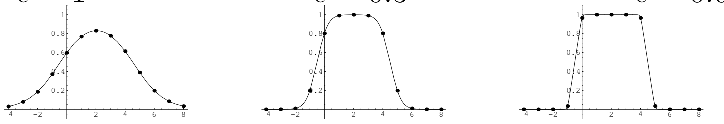
$\Omega = 3$

$\epsilon = 1$ $\epsilon = 0.3$ $\epsilon = 0.05$



$\Omega = 5$

$\epsilon = 1$ $\epsilon = 0.3$ $\epsilon = 0.05$



We have the BBB equation of the following form

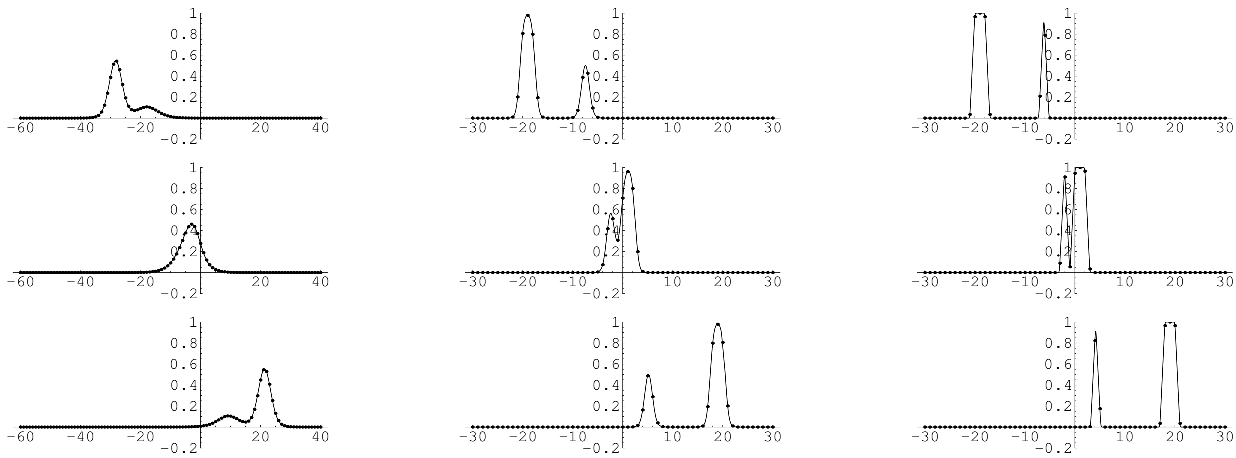
$$\begin{aligned} z_n^{m+1} &= 1 - \delta + \delta u_n^m u_{n-1}^m z_{n-1}^{m+1}, \\ u_{n+1}^{m+1} &= u_n^m z_n^{m+1} / z_{n+1}^{m+1}. \end{aligned}$$

Let $z_n^m = e^{Z_n^m/\epsilon}, u_n^m = e^{U_n^m/\epsilon}, \delta = e^{-1/\epsilon}$. Then

$$\begin{aligned} Z_n^{m+1} &= \epsilon \log[1 - \delta \exp[(U_n^m + U_{n-1}^m + Z_{n-1}^{m+1})/\epsilon], \\ U_n^{m+1} &= U_{n-1}^m + U_{n-1}^{m+1} - Z_n^{m+1}, \end{aligned}$$

which are used for maping of U_n^m. Following figures shows a collision of two solitons of magnitudes $\Omega_1 = 3$ and of $\Omega_2 = 1$.
Solitons are stable entities. They do not lose their identities after colliding with each another.

$\epsilon = 1$ $\epsilon = 0.3$ $\epsilon = 0.05$



3 Negative-Solitons

We have found numerically a negative-soliton of arbitrary magnitude moving with speed 1 satisfy the Box and Ball system

$$
\begin{aligned}
Z_n^{m+1} &= \max(0, U_n^m + U_{n-1}^m + Z_{n-1}^{m+1} - 1),\\
U_{n+1}^{m+1} &= U_n^m + Z_n^{m+1} - Z_{n+1}^{m+1}.
\end{aligned}
$$

The negative-solitons are expressed by $U_n^m = -P\delta(n-m)$, δ being Kronecker's delta. Let

$$f_n^m = 1 + p^{(n-m)}, \quad p = e^{P/\epsilon} \tag{7}$$

$$u_n^m = \frac{f_{n+1}^m f_n^{m+1}}{f_n^m f_{n+1}^{m+1}}. \tag{8}$$

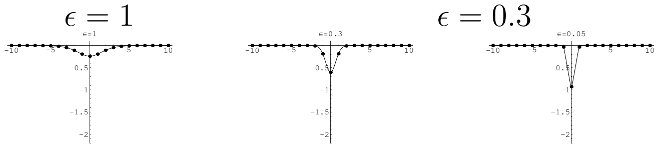
Then, negative-solitons are expressed by the ultradiscrete limit of $U_n^m(\epsilon)$,

$$U_n^m = \lim_{\epsilon \to 0} U_n^m(\epsilon) = -\lim_{\epsilon \to 0} \epsilon \log u_n^m.$$

Graphs of negative-solitons $U_n^m(\epsilon) = -\epsilon \log u_n^m$

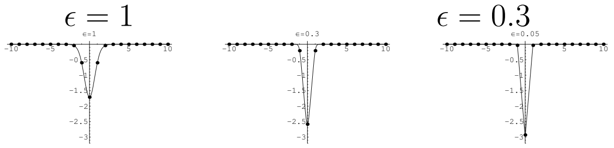
$P = 1$

$\epsilon = 1$ $\epsilon = 0.3$ $\epsilon = 0.05$



$P = 3$

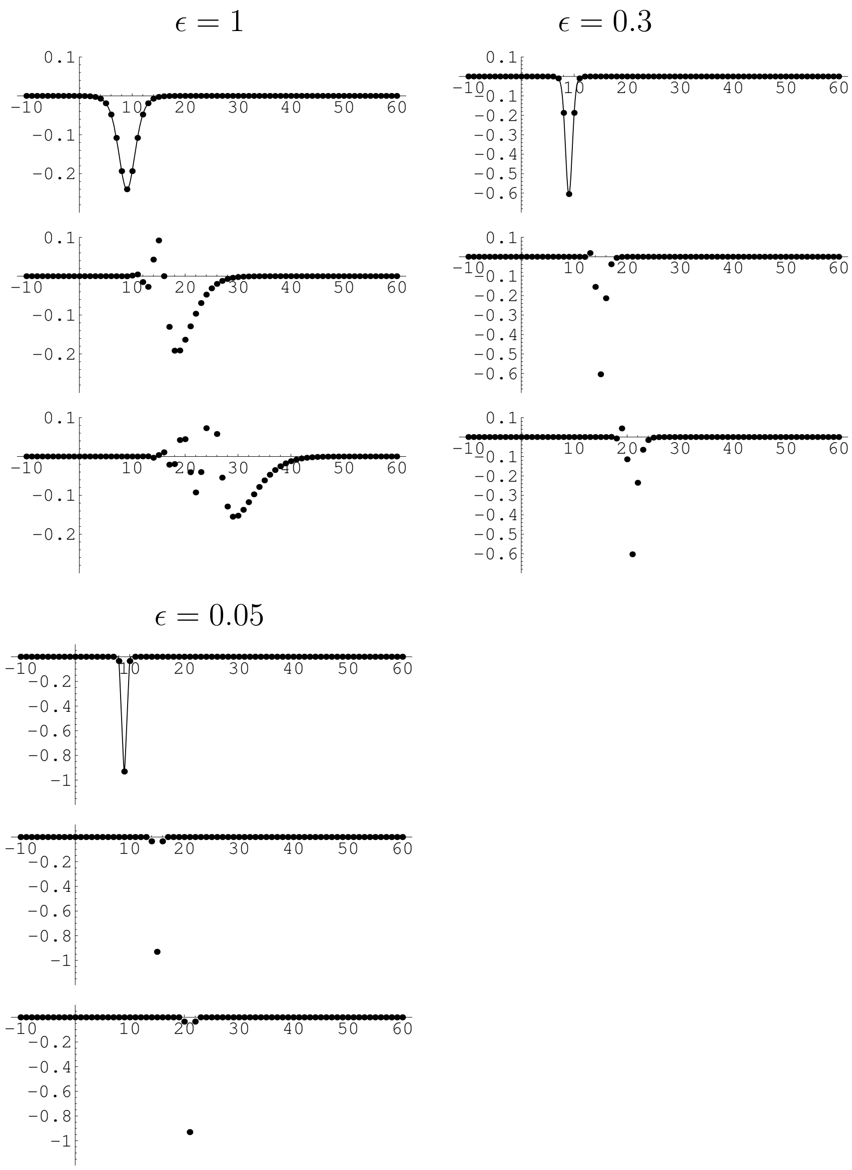
$\epsilon = 1$ $\epsilon = 0.3$ $\epsilon = 0.05$



We note that f_n^m of Eqs.(7) *does not* satisfy the bilinear equation

$$f_n^{m-1} f_{n+1}^{m+1} = \delta f_{n+1}^{m-1} f_n^{m+1} + (1-\delta) f_n^m f_{n+1}^m.$$

It is a solution to the bilinear equation only in the ultradiscrete limit.



The negative-solitons are not stable entities. They are stable only in the ultradiscrete limit.

We have the BBB equqation in the bilinear form,

$$f_n^{m-1} f_{n+1}^{m+1} = \delta f_{n+1}^{m-1} f_n^{m+1} + (1-\delta) f_n^m f_{n+1}^m,$$

which is reduced, in the ultradiscrete limit, to

$$F_n^{m-1} + F_{n+1}^{m+1} \\ = \max(F_{n+1}^{m-1} + F_n^{m+1} - 1, F_n^m + F_{n+1}^m).$$

Solutions to the Box and Ball system,$U_n^m,\ Z_n^m$ are given by the τ-functions F_n^m.

$$U_n^m = F_{n+1}^m + F_n^{m+1} - F_n^m - F_{n+1}^{m+1}, \tag{9}$$
$$Z_n^m = F_n^{m-1} + F_{n+1}^{m+1} - F_n^m - F_{n+1}^m. \tag{10}$$

4 τ-functions

We look for τ-functions describing a collision of a soliton(balls) with a negative-soliton. We know that the τ-function of a soliton(balls) is given,in the ultradiscrete limit, by

$$\begin{aligned} F_n^m &= \max(0, r_1(m,n)), \\ r_1(m,n) &= \Omega_1 m + P_1 n, \\ P_1 &= \max(0, \Omega_1 - 1) - \max(0, -\Omega_1 - 1). \end{aligned}$$

We have found that τ-function $h(m,n)$ expressing a negative-soliton is given by

$$h(m,n) = h_1 \min(0, m-n), \tag{11}$$

where h_1 is an arbitrary constant expressing the magnitude of negative-soliton.
We find using Eq.(9)

$$\begin{aligned} U_n^m &= h_1[\min(0, m-n+1) \\ &\quad + \min(0, m-n-1) - 2\min(0, m-n)], \\ &\leq 0. \end{aligned} \tag{12}$$

U_n^m is negative because that U_n^m is a second difference of a convex function.

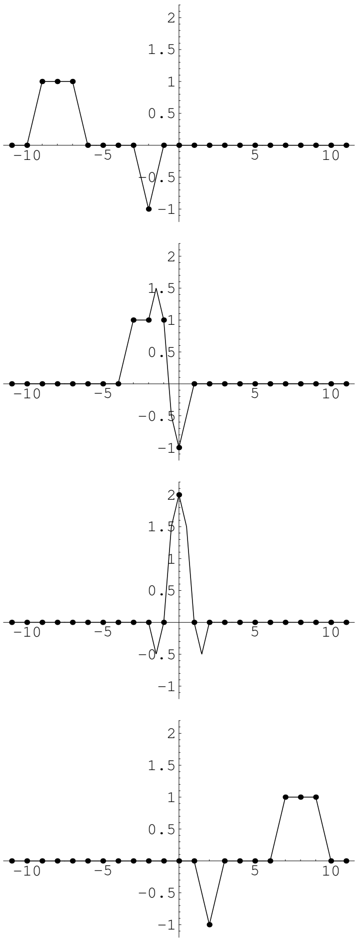
We conjecture that τ-function F_n^m describing a collision of a soliton (balls) with a negative-soliton is given by

$$F_n^m = \max[h(m,n), r_1(m,n) + h(m,n+2)].$$

In the following figures, solid lines express theoretical solutions U_n^m as a function of a continuous variable n. While the dots indicate numerical values of U_n^m calculated by the Box and Ball system for integers n.

The parameters used are $\Omega_1 = 3, h_1 = 1$.

All dots are on the solid lines, which strongly suggest that the conjectured τ-function is correct.



5 Proof of the conjecture

We prove that the τ-function,

$$F_n^m = \max[h(m,n), r_1(m,n) + h(m,n+2)] \tag{13}$$

satisfies the ultradiscrete bilinear equation,

$$F_n^{m-1} + F_{n+1}^{m+1} = \max(F_{n+1}^{m-1} + F_n^{m+1} - 1, F_n^m + F_{n+1}^m), \tag{14}$$

where

$$\begin{aligned} h(m,n) &= h_1 \min(0, m-n), \quad h_1 > 0, \\ r_1 &= r_1(m,n) = \Omega_1 m + P_1 n, \quad \Omega_1 > 0, \\ P_1 &= \max(0, \Omega_1 - 1) - \max(\Omega, -1). \end{aligned}$$

We calculate the *l.h.s* of Eq.(14)

$$
\begin{aligned}
&F_n^{m-1} + F_{n+1}^{m+1} \\
&\quad = \max(h(m-1,n), r_1 - \Omega_1 + h(m-3,n)) \\
&\quad + \max(h(m,n), r_1 + \Omega_1 + P_1 + h(m-2,n)), \\
&\quad = \max(h(m-1,n) + h(m,n), \\
&\qquad r_1 + \Omega_1 + P_1 + h(m-2,n) + h(m-1,n), \\
&\qquad r_1 - \Omega_1 + h(m-3,n) + h(m,n), \\
&\qquad 2r_1 + P_1 + h(m-3,n) + h(m-2,n)), \\
&\quad = \max(\phi_0^a, r_1(m,n) + \phi_1^a, 2r_1(m,n) + \phi_2^a),
\end{aligned}
$$

where

$$
\begin{aligned}
\phi_0^a &= h(m-1,n) + h(m,n), \\
\phi_1^a &= \max(\Omega_1 + P_1 + h(m-2,n) + h(m-1,n), \\
&\qquad -\Omega_1 + h(m-3,n) + h(m,n)), \\
\phi_2^a &= P_1 + h(m-3,n) + h(m-2,n).
\end{aligned}
$$

Similarly we find that

$$
F_{n+1}^{m-1} + F_n^{m+1} = \max(\phi_0^b, r_1(m,n) + \phi_1^b, 2r_1(m,n) + \phi_2^b),
$$

where

$$
\begin{aligned}
\phi_0^b &= h(m-2,n) + h(m+1,n), \\
\phi_1^b &= \max(\Omega_1 + h(m-2,n) + h(m-1,n), \\
&\qquad -\Omega_1 + P_1 + h(m-4,n) + h(m+1,n)), \\
\phi_2^b &= P_1 + h(m-4,n) + h(m-1,n),
\end{aligned}
$$

and

$$
F_n^m + F_{n+1}^m = \max(\phi_0^c, r_1(m,n) + \phi_1^c, 2r_1(m,n) + \phi_2^c),
$$

where

$$
\begin{aligned}
\phi_0^c &= h(m-1,n) + h(m,n), \\
\phi_1^c &= \max(P_1 + h(m-3,n) + h(m,n), \\
&\qquad\qquad h(m-2,n) + h(m-1,n)), \\
\phi_2^c &= P_1 + h(m-3,n) + h(m-2,n).
\end{aligned}
$$

Substituting these expressions into the ultradiscrete bilinear equation (14) we obtain

$$\begin{aligned}\max(\phi_0^a, r_1+\phi_1^a, 2r_1+\phi_2^a) = \max(&\max(\phi_0^b, r_1+\phi_1^b, 2r_1+\phi_2^b)-1,\\ &\max(\phi_0^c, r_1+\phi_1^c, 2r_1+\phi_2^c)),\end{aligned}$$

which gives the sufficient conditons for F_n^m to be a solution to Eq.(14),

$$\phi_j^a = \max(\phi_j^b - 1, \phi_j^c), \text{for } j = 0, 1, 2. \tag{15}$$

We prove Eqs.(15) for $j = 0$ and 2 first.
Substracting ϕ_j^a we obtain,

$$\max(\phi_j^b - \phi_j^a - 1, \phi_j^c - \phi_j^a) = 0, \text{for } j = 0, 2. \tag{16}$$

We find

$$\phi_0^c = \phi_0^a, \quad \phi_2^c = \phi_2^a$$

and

$$\begin{aligned}\phi_0^b - \phi_0^a &= h(m-2,n) - h(m-1,n) - h(m,n) + h(m+1,n) \le 0,\\ \phi_2^b - \phi_2^a &= h(m-4,n) - h(m-3,n) - h(m-2,n) + h(m-1,n) \le 0.\end{aligned}$$

Hence the conditions(16) for $j = 0$ and 2 are satisfied.
We introduce new variable $V_h(m,n)$ in order to prove the conditions(16) for $j = 1$

$$V_h(m,n) = h(m-2,n) - h(m-1,n) - h(m,n) + h(m+1,n) \le 0, \tag{17}$$

which gives a relation

$$\begin{aligned}&h(m-4,n) - h(m-2,n) - h(m-1,n) + h(m+1,n)\\ &= V_h(m-1,n) + V_h(m,n) + V_h(m+1,n).\end{aligned} \tag{18}$$

Using Eqs.(17) and (18) we find

$$\phi_1^a(n) = h(m-2,n) + h(m-1,n) + \max(\Omega_1 + P_1, -\Omega_1 + V_h(m,n)),$$

$$\begin{aligned}&\phi_1^b(n) = h(m-2,n) + h(m-1,n)\\ &+\max(\Omega_1, -\Omega_1 + P_1 + V_h(m-1,n) + V_h(m,n) + V_h(m+1,n)),\end{aligned}$$

and

$$\phi_1^c(n) = h(m-2,n) + h(m-1,n) + \max(P_1 + V_h(m,n), 0).$$

Subtracting $h(m-2,n) + h(m-1,n)$ from Eq.(16) for $j = 1$ we obtain

$$\begin{aligned}
&\max(\Omega_1 + P_1, -\Omega_1 + V_h(m,n)) \\
&\quad = \max(\Omega_1 - 1, -\Omega_1 + P_1 + V_h(m-1,n) + V_h(m,n) + V_h(m+1,n) - 1, \\
&\quad P_1 + V_h(m,n), 0),
\end{aligned}$$

We have for $0 \le \Omega_1$

$$P_1 = \begin{cases} -1, & \text{for } 1 \le \Omega_1 \\ -\Omega_1, & \text{for } 0 \le \Omega_1 < 1 \end{cases}$$

and

$$V_h(m-1,n), V_h(m,n), V_h(m+1,n) \le 0$$

Hence the conditions(16) for $j = 1$,

$$\phi_1^a = \max(\phi_1^b - 1, \phi_1^c)$$

is reduced to

$$\Omega_1 + P_1 = \max(\Omega_1 - 1, 0)$$

which holds for $0 \le \Omega_1$.
We have proved that the conditions(16) hold for $j = 1, 2, 3$. Therefore F_n^m of Eq.(13) satisfies the ultradiscrete bilinear equation (14).