low temperature limit of equations
— hidden discrete structure

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“ultradiscretization method” found in the area of soliton theory

introduction to soliton (integrable) equations

N soliton solution, infinite number of conserved quantities, bilinear form, inverse scattering method, Lax form, Sato theory

$u_t + 6uu_x + u_{xxx} = 0$	Korteweg–de Vries (KdV) eq
$(u_t + 6uu_x + u_{xxx})_x \pm u_{yy} = 0$	Kadomtsev–Petviashvili (KP) eq
$i\psi_t + \psi_{xx} + \lvert\psi\rvert^2\psi = 0$	nonlinear Schrödinger eq
$u_{xt} = \sin u$	sine-Gordon eq
$u_{tt} - c^2 u_{xx} - (u^2)_{xx} - u_{xxxx} = 0$	Boussinesq eq
$\begin{cases} iu_t - u_{xx} + u_{yy} + u\lvert u\rvert^2 - 2uv = 0 \\ v_{xx} + v_{yy} - (\lvert u\rvert^2)_{yy} = 0 \end{cases}$	Davey–Stewartson eq
$\dfrac{d^2 x_j}{dt^2} = 2\sum_{k\neq j} \dfrac{1}{(x_j - x_k)^3}$	Calogero system
$\dfrac{d^2 y_n}{dt^2} = e^{-b(y_n - y_{n-1})} - e^{-b(y_{n+1} - y_n)}$	Toda lattice eq
$\dfrac{du_n}{dt} = u_n(u_{n+1} - u_{n-1})$	Lotka–Volterra eq

Korteweg–de Vries eq

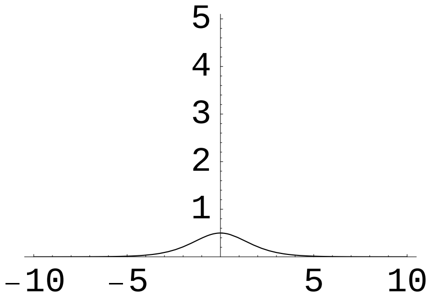
$$\boxed{u_t + 6uu_x + u_{xxx} = 0}$$

$$\updownarrow \qquad u(x,t) = 2(\log f(x,t))_{xx}$$

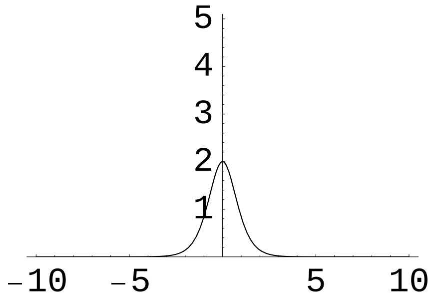
$$\boxed{ff_{xt} - f_x f_t + ff_{4x} - 4f_x f_{xxx} + 3f_{xx}^2 = 0} \qquad \text{(bilinear form)}$$

one soliton sol

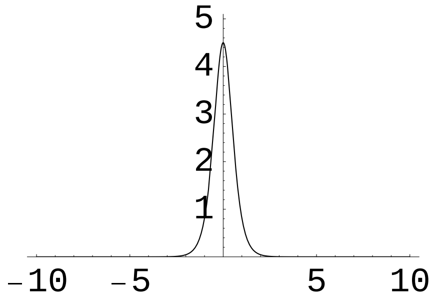
$$f(x,t) = 1 + \exp(kx - k^3 t + c), \qquad u(x,t) = \frac{k^2}{2} \frac{1}{\cosh^2 \frac{kx - k^3 t + c}{2}}$$



$k = 1$



$k = 2$

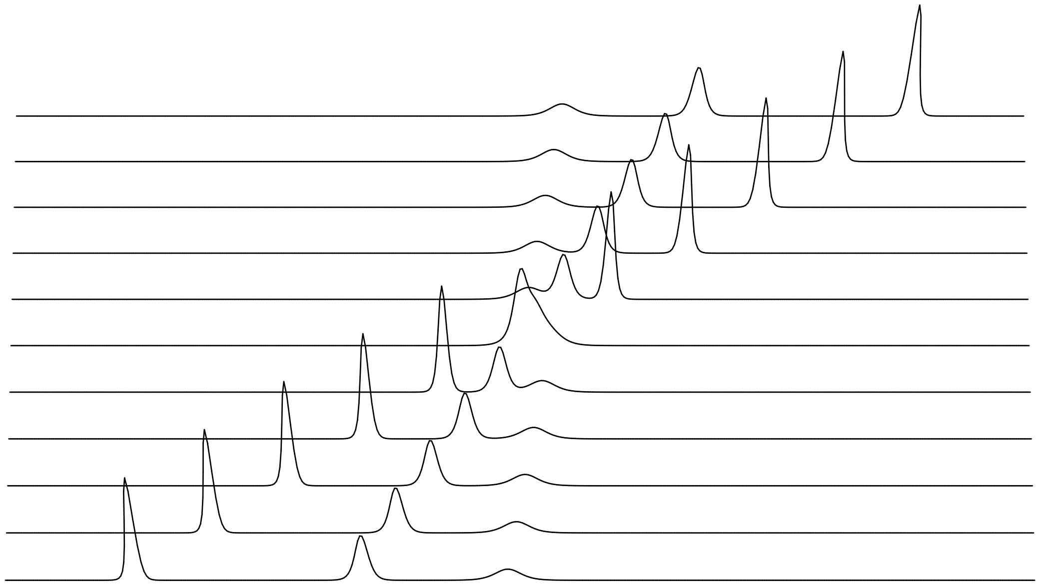


$k = 3$

three soliton sol

$$f(x,t) = 1 + \exp(\eta_1) + \exp(\eta_2) + \exp(\eta_3) + a_{12}\exp(\eta_1+\eta_2) + a_{13}\exp(\eta_1+\eta_3) + a_{23}\exp(\eta_2+\eta_3) + a_{12}a_{13}a_{23}\exp(\eta_1+\eta_2+\eta_3)$$

where $\eta_j(x,t) = k_j x - k_j^3 t + c_j,\qquad a_{ij} = \dfrac{(k_i-k_j)^2}{(k_i+k_j)^2}$



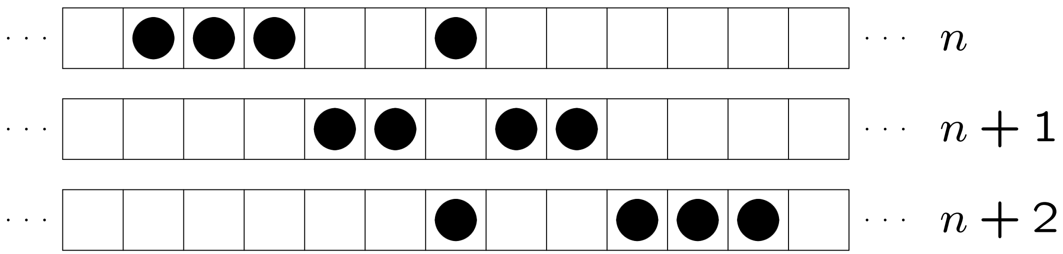
$k_1 = 1,\ k_2 = 2,\ k_3 = 3,\ c_1 = c_2 = c_3 = 0$

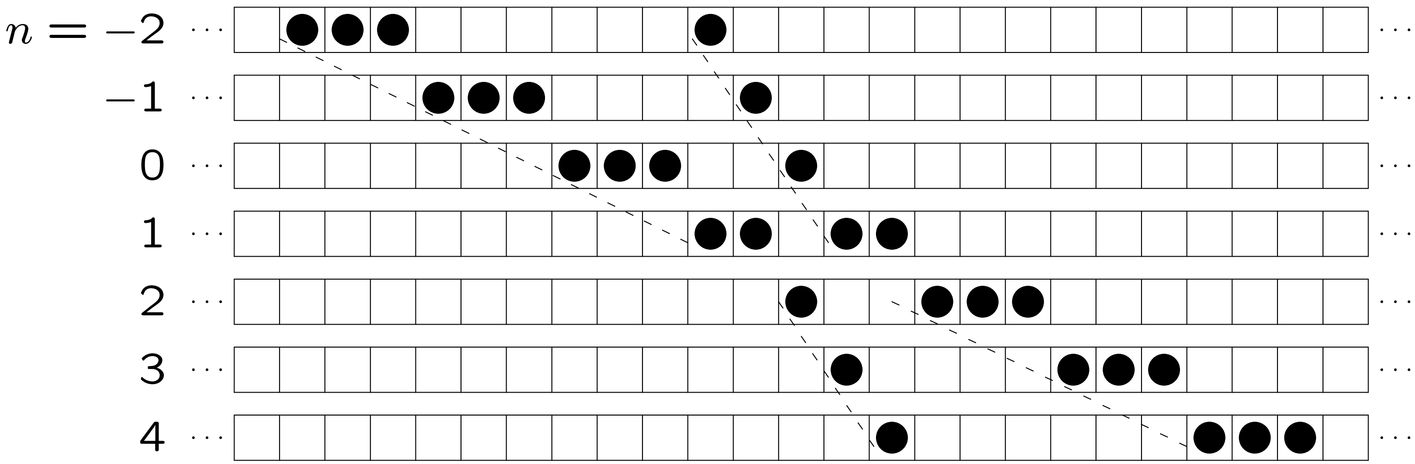
pulses behave like particles : solitary wave + on (suffix meaning ‘particle’)

digital soliton system

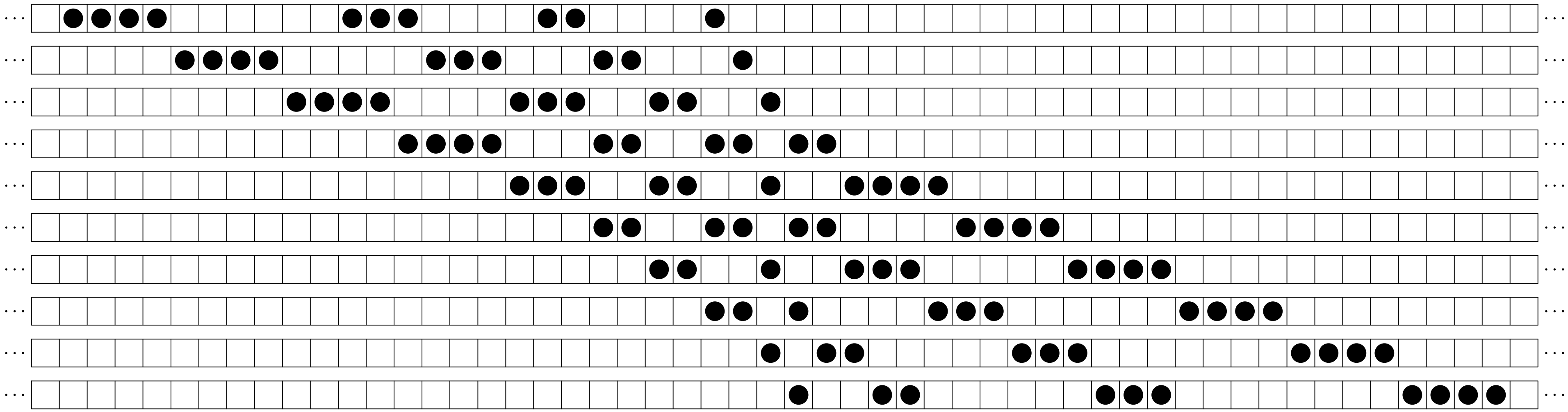
(T&Satsuma, J. Phys. Soc. Jpn. **59** (1990) 3514–3519)

box and ball system (BBS) — balls move in a line of boxes





from any initial state, soliton behavior obtained



All variables are integer: time, space, state value

$\Rightarrow$ What is a relation to known soliton systems?

Can we derive it from a known soliton eq?

(e.g. $u_t + 6uu_x + u_{xxx} = 0$)

Expression of evolution rule of BBS

B_j^n $(= 0$ or $1)$: number of ball in the j-th box at time n

$$B_j^{n+1} = \min(1 - B_j^n, \sum_{i=-\infty}^{j-1} (B_i^n - B_i^{n+1}))$$

$$\Downarrow \quad S_j^n = \sum_{i=-\infty}^{j} B_j^n, \quad V_j^n = S_{j+1}^n - S_j^{n+1}, \quad V_j^n = U_{n-j}^{n+j}$$

$$\underbrace{U_j^{n+1} - U_j^{n-1}}_{\text{1st order}} = \underbrace{\max(0, U_{j+1}^n - 1) - \max(0, U_{j-1}^n - 1)}_{\text{1st order}}$$

Known soliton eq

1st+1st : $\dfrac{dv_j}{dt} = e^{v_{j-1}} - e^{v_{j+1}}$ (Lotka–Volterra eq)

2nd+2nd : $\dfrac{d^2 v_j}{dt^2} = e^{v_{j+1}} - 2e^{v_j} + e^{v_{j-1}}$ (Toda lattice eq)

$\Rightarrow$ **How about 2nd order of BBS?**

2nd order digital soliton eq

```
...24............3..71
....33..........21.62.
.....42........12.53..
......51.......3.44...
.......6......2135....
.......15....1226.....
........24...317......
.........33.218.......
..........43271.......
..........1682........
..........3662........
.........263.51.......
........1721..6.......
........812...15......
.......713.....24.....
......6221......33....
.....5312........42...
....44.3..........51..
...35.21...........6..
..26.12............15.
.17..3..............24
.8..21...............3
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Ultradiscrete Toda lattice eq

$$U_j^{n+1} - 2U_j^n + U_j^{n-1} = \max(0, U_{j+1}^n - 1) - 2\max(0, U_j^n - 1) + \max(0, U_{j-1}^n - 1)$$

R.Hirota: "Nonlinear Partial Difference Equations II—Discrete-Time Toda Equation"

J. Phys. Soc. Jpn. **43** (1977) 2074–2078

$$\frac{d^2 v_j}{dt^2} = e^{v_{j+1}} - 2e^{v_j} + e^{v_{j-1}}$$ (Toda lattice eq)

$\Uparrow \quad v_j(\delta n) = u_j^n, \quad \delta \to 0$ (continuum limit)

$$u_j^{n+1} - 2u_j^n + u_j^{n-1}$$
$$= \log(1+\delta^2(e^{u_{j+1}^n}-1)) - 2\log(1+\delta^2(e^{u_j^n}-1)) + \log(1+\delta^2(e^{u_{j-1}^n}-1))$$

. . . .

$\Downarrow \quad u_j^n = U_j^n/\varepsilon, \quad \delta = e^{-1/2\varepsilon}, \quad \varepsilon \to +0$ (ultradiscrete limit)

formula : $\lim_{\varepsilon\to+0} \varepsilon \log(e^{A/\varepsilon} + e^{B/\varepsilon}) = \max(A, B)$

$$U_j^{n+1} - 2U_j^n + U_j^{n-1}$$
$$= \max(0, U_{j+1}^n - 1) - 2\max(0, U_j^n - 1) + \max(0, U_{j-1}^n - 1)$$

(ultradiscrete Toda lattice eq)

Key formula

$$\lim_{\varepsilon \to +0} \varepsilon \log(e^{A/\varepsilon} + e^{B/\varepsilon}) = \max(A, B)$$

statistical mechanics :

a state of minimum free energy appears in the low temperature limit

$$\lim_{T \to +0} -kT \log(e^{-E_1/kT} + e^{-E_2/kT} + \cdots) = \min(E_1, E_2, \cdots)$$

“low temperature limit” of equation digitizes a variable

full scenario from KdV eq to BBS

$$w_t + 6ww_x + w_{xxx} = 0$$ KdV eq

$\uparrow \quad v_j(t) = 1 + \epsilon^2 w(\epsilon(j-2t), \frac{\epsilon^3}{3}t), \quad \epsilon \to 0$ continuum limit

$$\frac{dv_j}{dt} = v_j(v_{j-1} - v_{j+1})$$ Lotka–Volterra eq

$\uparrow \quad v_j(n\delta) = u_j^n, \quad \delta \to 0$ continuum limit

$$\frac{1}{\delta}(u_j^{n+1} - u_j^n) = u_j^n u_{j-1}^n - u_j^{n+1} u_{j+1}^{n+1}$$ discrete LV eq

$\downarrow \quad u_j^n = \exp(U_j^n/\varepsilon), \quad \delta = \exp(-1/\varepsilon), \quad \varepsilon \to +0$ ultradiscretization

$$U_j^{n+1} - U_j^n = \max(0, U_{j-1}^n - 1) - \max(0, U_{j+1}^{n+1} - 1)$$ ultradiscrete LV eq

$\uparrow \quad V_j^n = U_{j-n}^j$ coordinate trans, $\quad V_j^n = S_{j+1}^n - S_j^{n+1}, \quad S_j^n = \sum_{i=-\infty}^{j} T_i^n$ variable trans

$$B_j^{n+1} = \min(1 - B_j^n, \sum_{i=-\infty}^{j-1}(B_i^n - B_i^{n+1}))$$ BBS

N soliton solutions can also be transformed

Ultradiscretization of the Burgers eq

$$v_t = 2vv_x + v_{xx}$$ (Burgers eq)

$\uparrow$ $v(j\Delta x, n\Delta t) = u_j^n, \quad \Delta x, \Delta t \to 0$ (discrete Burgers eq)

$$u_j^{n+1} = u_j^n + \frac{1}{\Delta x}\Big\{\log(e^{-\Delta x\, u_j^n} + e^{\Delta x\, u_{j+1}^n}) - \log(e^{-\Delta x\, u_{j-1}^n} + e^{\Delta x\, u_j^n})\Big\}$$

$\downarrow$ $\Delta x \cdot u_j^n = (U_j^n - \frac{1}{2})/\varepsilon, \quad \varepsilon \to +0$ (ultradiscrete Burgers eq)

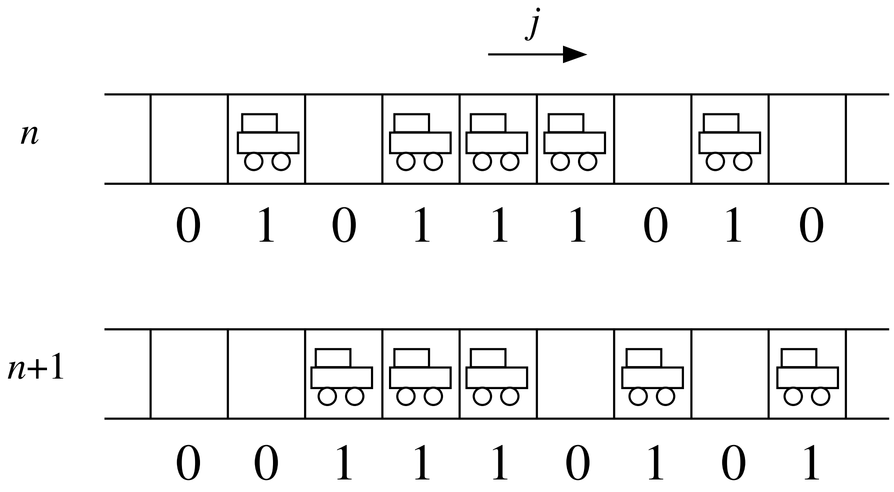
$$U_j^{n+1} = U_j^n + \min(U_{j-1}^n,\, 1 - U_j^n) - \min(U_j^n,\, 1 - U_{j+1}^n)$$

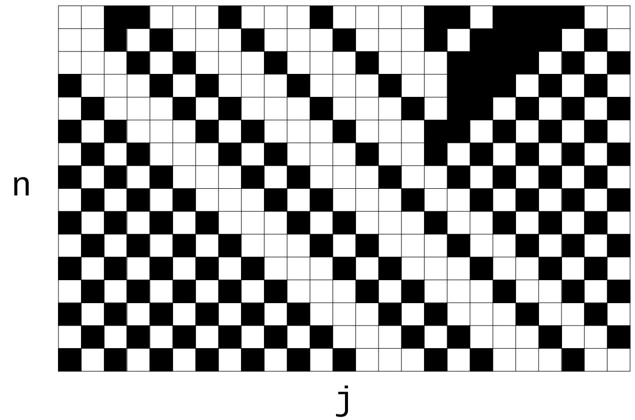
$\downarrow$ $U_j^0 \in \{0, 1\} \ \rightarrow \ U_j^n \in \{0, 1\}$ for any $n > 0$ (Burgers CA)

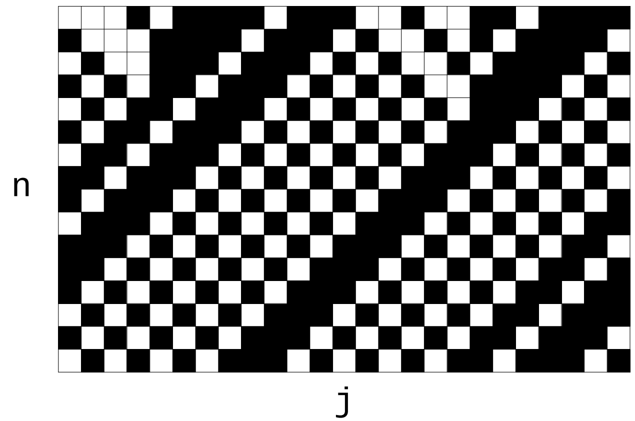
$$\frac{U_{j-1}^n\, U_j^n\, U_{j+1}^n}{U_j^{n+1}} : \quad \frac{000}{0} \quad \frac{001}{0} \quad \frac{010}{0} \quad \frac{011}{1} \quad \frac{100}{1} \quad \frac{101}{1} \quad \frac{110}{0} \quad \frac{111}{1}$$

Burgers CA is a toy model of traffic flow

$$\frac{U_{j-1}^n \, U_j^n \, U_{j+1}^n}{U_j^{n+1}} : \quad \frac{000}{0} \quad \frac{001}{0} \quad \frac{010}{0} \quad \frac{011}{1} \quad \frac{100}{1} \quad \frac{101}{1} \quad \frac{110}{0} \quad \frac{111}{1}$$







ultradiscrete eq & max–plus algebra

$$\lim_{\varepsilon\to+0}\varepsilon\log(e^{A/\varepsilon}+e^{B/\varepsilon})=\max(A,B),\qquad \lim_{\varepsilon\to+0}\varepsilon\log(e^{A/\varepsilon}-e^{B/\varepsilon})=???$$

$$(\lim_{\varepsilon\to+0})\varepsilon\log(e^{A/\varepsilon}\times e^{B/\varepsilon})=A+B,\qquad (\lim_{\varepsilon\to+0})\varepsilon\log(e^{A/\varepsilon}/\,e^{B/\varepsilon})=A-B$$

	$(+,\times)$	$(\max,+)$
commutative law	$a+b=b+a$ $a\times b=b\times a$	$\max(A,B)=\max(B,A)$ $A+B=B+A$
associative law	$a+(b+c)$ $=(a+b)+c$ $a\times(b\times c)$ $=(a\times b)\times c$	$\max(A,\max(B,C))$ $=\max(\max(A,B),C)$ $A+(B+C)$ $=(A+B)+C$
distributive law	$a\times(b+c)$ $=a\times b+a\times c$	$A+\max(B,C)$ $=\max(A+B,A+C)$

$$\boxed{a(b+c)=\frac{d+e}{f+g}}\ \xrightarrow[\varepsilon\to+0]{a=e^{A/\varepsilon},\ \cdots}\ \boxed{A+\max(B,C)=\max(D,E)-\max(F,G)}$$

difference eq

max–plus eq (piecewise linear eq)

‘Soliton’ is not necessary for ultradiscretization

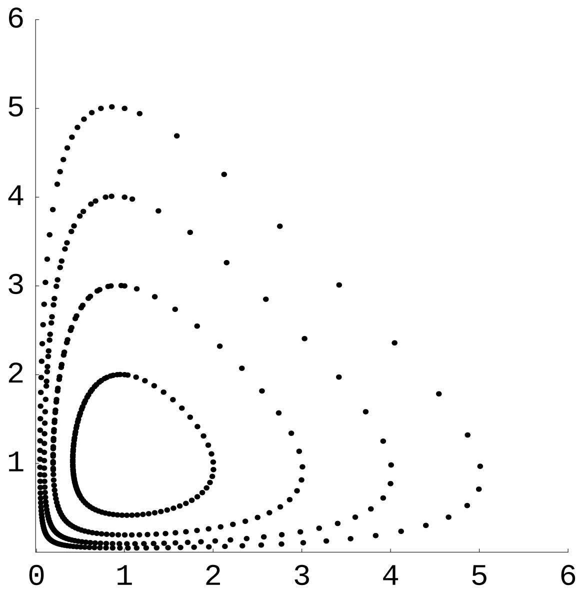
Application to non-solitonic equations

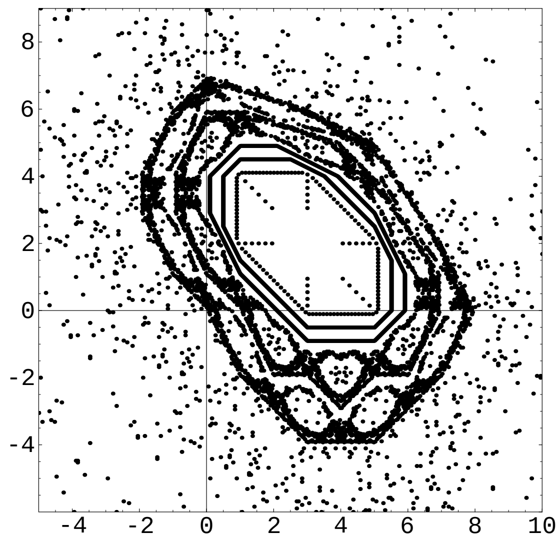
$\Rightarrow$ Reveal a hidden discrete structure of a continuous system!

ex1: chaos

$$\begin{cases} \dfrac{1}{\delta}(x_{n+1} - x_n) = a x_n - x_{n+1} y_{n+1} \\ \dfrac{1}{\delta}(y_{n+1} - y_n) = x_n y_n - b y_{n+1} \end{cases} \to \begin{cases} X_{n+1} = X_n + A - \max(0, Y_{n+1}) \\ Y_{n+1} = Y_n + \max(0, X_n) - B \end{cases}$$

$$x_n = e^{X_n/\varepsilon}, \quad y_n = e^{Y_n/\varepsilon}, \quad 1 + a = e^{A/\varepsilon}, \quad 1 + b = e^{B/\varepsilon}, \quad \varepsilon \to +0$$





exaggeration of a solution reveals its structure

ex2: attractor

$$\boxed{X_{n+1} = \max(0, X_{n-2}) - X_n}$$

period 5 appears from any initial data

$$1,\ 2,\ 3,\ -2,\ \underline{4,\ -1,\ 1,\ 3,\ -3},\ \underline{4,\ -1,\ 1,\ 3,\ -3},\ \cdots$$
$$3,\ 2,\ 1,\ 2,\ 0,\ \underline{1,\ 1,\ -1,\ 2,\ -1},\ \underline{1,\ 1,\ -1,\ 2,\ -1},\ \cdots$$

Lyapunov function ($L_n \geq L_{n+1}$, $L_n = 0 \leftrightarrow$ period 5)

$$\boxed{L_n = |\min(X_{n-2}, X_n, X_{n-2} + X_n) + X_{n-1}|}$$

....

parallel world $\quad (x_n = e^{X_n/\varepsilon})$

$$\boxed{x_{n+1} = \frac{1 + x_{n-2}}{x_n}}$$

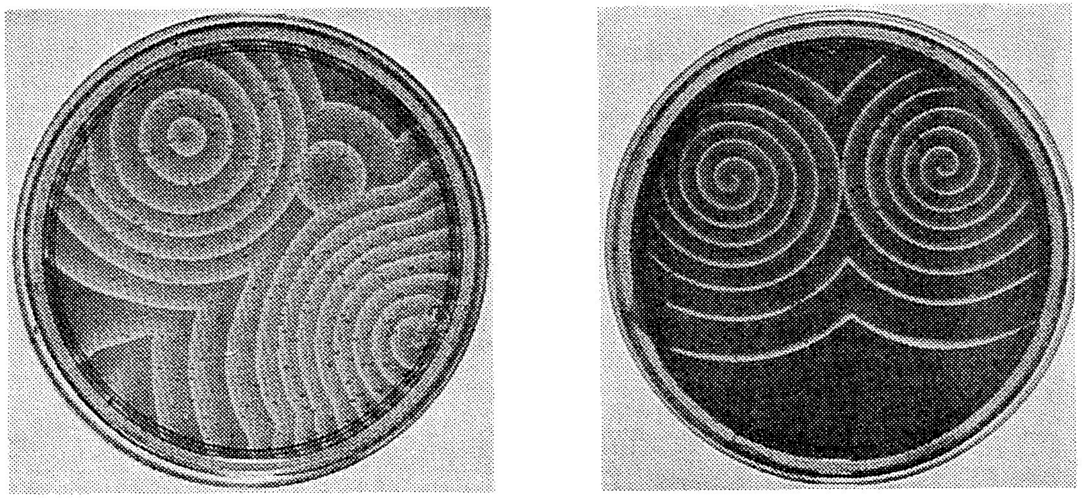
Lyapunov function ($\ell_n > \ell_{n+1}$, $\ell_n \to 2 \leftrightarrow$ attractor)

$$\boxed{\ell_n = \frac{x_{n-2}x_{n-1}x_n}{1 + x_{n-2} + x_n} + \frac{1 + x_{n-2} + x_n}{x_{n-2}x_{n-1}x_n}}$$

appearance is very different, but structure is very similar

ex3: pattern formation system

Belouzov–Zhabotinsky (BZ) reaction (chemical reaction in a petri dish)



(a) (b)

model eq

$$\begin{cases} u_t = D_u(u_{xx} + u_{yy}) + \dfrac{1}{\delta}\{u(1-u) - \dfrac{F\,v\,(u-q)}{u+q}\} \\ v_t = D_v(v_{xx} + v_{yy}) + u - v \end{cases}$$

demo: numerical simulation

max–plus "TOY" model

$$\begin{cases} U_{ij}^{n+1} = \max(U_{ij}^n, U_{i-1j}^n, U_{i+1j}^n, U_{ij-1}^n, U_{ij+1}^n) - V_{ij}^n, \\ V_{ij}^{n+1} = U_{ij}^n \end{cases}$$

$$\Updownarrow$$

$$\boxed{U_{ij}^{n+1} = \max(U_{ij}^n, U_{i-1j}^n, U_{i+1j}^n, U_{ij-1}^n, U_{ij+1}^n) - U_{ij}^{n-1}}$$

symmetric in space & time

demo: numerical simulation

Is there a very simple PDE corresponding to the above?

$\Rightarrow$ I strongly hope to know the answer using ultradiscretization

concluding remarks

- Ultradiscretization reveals a hidden discrete structure of continuous equations.
- It can be applied to non-integrable (non-solitonic) systems.
- We may connect differential equations and digital processes more directly by the ultradiscretization.