A Stochastic Model for Solitons

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Abstract. The soliton physics for the propagation of waves is represented by a stochastic model in which the particles of the wave can jump ahead according to some probability distribution. We demonstrate the presence of a steady state (stationary distribution) for the wavelength. It is shown that the stationary distribution is a convolution of geometric random variables. Approximations to the stationary distribution are investigated for a large number of particles. The model is rich and includes Gaussian cases as limit distribution for the wavelength (when suitably normalized). A sufficient Lindeberg-like condition identifies a class of solitons with normal behavior. Our general model includes, among many other reasonable alternatives, an exponential aging soliton, of which the uniform soliton is one special subcase (with Gumbel's stationary distribution). With the proper interpretation, our model also includes the deterministic model proposed in Takahashi and Satsuma (1990).

Keywords: Soliton, wave propagation, random structure, limit distribution.

1 Introduction

Wave motion is a very important phenomenon in physics, chemistry, electrical engineering, etc. It is a key notion related to propagation, change of state, and pattern formation in various nonlinear systems. Many nonlinear dynamical models are proposed to simulate such phenomena. To grasp the rudimental dynamics in such phenomena, both simplicity and refinement are necessary for the model. Recently discrete models, for example cellular automata, have become popular because of the ease of their analysis and simulation; Wolfram (1986) provides a survey of cellular automata. However, simplicity and refinement are sometimes opposing goals that cannot be reconciled. Stochasticity can be effectively used to resolve this incompatibility. Previously proposed soliton

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models mostly focused on local interaction among particles (Park, Steiglitz and Thurston (1986)). In this paper, we propose a stochastic discrete model for solitonical wave motion that takes into account the global interaction among particles extending the deterministic model by Takahashi and Satsuma (1990). The model is governed by quite simple rules.

Our present study is on a stochastic cellular automaton for *one* solitonical wave. Soliton collisions are observed in the deterministic model by Takahashi and Satsuma (1990) as in the solution of the Kordwerg de Vries equation which is a partial differential equation for solitonical waves. Extension of our present model to interacting solitonical waves will be a future research project. The study relates to the wider field of nonlinear integrable dynamical systems and their stochastic models (Arnold (1989), Itoh (1979, 1993)). The relation to these models will be also a future research project.

Takahashi and Satsuma (1990) present a toy model for solitons. We extend this model to a richer stochastic toy model with a large variety of subcases. We model a unidirectional wave by a stochastic system of particles moving in one direction. Our model may shed some light on the movement of shallow-water waves and similar physical systems. The wave is assumed to be initially a system of k particles, aligned in a linear configuration, say in straight horizontal line segment, without gaps in between. In a real soliton system, such as a shallow-water wave, k is very large. The particles are indistinguishable. Regardless of history, the particles are always numbered, say, $1, \ldots, k$, from left to right. The wave propagates to the left—one of the particles jumps forward, ahead of the first particle (the *wave front*), and takes a position immediately to its left. When the jth particle jumps forward, it leaves behind an empty space, if $j = 1, \ldots, k-1$. If the last particle is the one that jumps, the penultimate particle becomes the last one. The process repeats indefinitely in discrete time steps $n = 1, 2, \ldots$. After $n-1$ stages we have a configuration of particles and gaps, in which the particles are numbered from left to right $1, \ldots, k$. The nth configuration evolves from the $(n-1)$st by having one of the particles jump forward, leaving a gap behind. If the particle moving forward is the last in the wave, the penultimate particle in configuration $n-1$ becomes the last in configuration n. Any spaces between these two particles appear now to the right of the rightmost particle, and are thus not part of the wave any more. Right after the $(n-1)$st jump, we *renumber* the particles in the nth configuration $1, \ldots, k$ from left to right. Many particles' labels may change from one configuration to the next. See Figure 1 for one possible evolution of the wave, and note the renumbering scheme between configurations. In the figure the down arrow indicates the particle that will jump next.

We are interested in knowing whether this system reaches a state of equilibrium, and the position of the particles in this system in the long run, in particular the distance of the last particle relative to the wave front (which defines the length of the wave).

The paper is organized as follows. Section 2 sets up a stochastic soliton model under a general distribution for jumping. Section 3 establishes the existence of a steady state distribution, which is also explicitly characterized as a convolution of geometric distributions. In Section 4 a Lindeberg-like condition is discussed and illustrated by one example. In Section 5 an exponentially aging soliton is considered: The degenerate uniform case is taken up in Subsection 5.1, and the bona fide nondegenerate aging scheme is taken up in Subsection 5.2.

In the sequel we use standard probability notation: Geo(β) stands for the geometric random variable with rate of success β per experiment, and $\mathcal{N}(\mu, \sigma^2)$ denotes the normally distributed random variate with mean μ and variance σ^2. The notation $\xrightarrow{\mathcal{D}}$ denotes convergence in distribution.

Several results are conveniently stated in terms of the mth degree harmonic number of order j: $H_j^{(m)} = \sum_{i=1}^{j} 1/i^m$. Asymptotics for these results are obtained from the following well-known approximations, as $n \to \infty$,

$$\begin{aligned} H_n^{(1)} &= \ln n + \gamma + \frac{1}{2n} + O\Big(\frac{1}{n^2}\Big), \\ H_n^{(2)} &= \frac{\pi^2}{6} - \frac{1}{n} + O\Big(\frac{1}{n^2}\Big), \end{aligned}$$

where $\gamma = 0.577215\ldots$ is Euler's constant.

2 Mathematical modeling

At any stage, we assume the jth particle jumps forward according to a *jumping probability distribution* p_j. Thus, necessarily $p_1 + \cdots + p_k = 1$. This stochastic model covers a variety of practical systems, such as an exponential aging system, in which particles that did not jump for a while are given more chance to catch up:

$$p_j = \frac{r^{j-1}}{r + \cdots + r^{k-1}}, \qquad \text{for} \quad j \geq 2, \qquad\qquad p_1 = 0, \tag{1}$$

for some $r \geq 1$. The special case $r = 1$ gives a uniform distribution on the particles that are not on the wave front. The case $r = \infty$ gives the deterministic model proposed by Takahashi and Satsuma (1990), in which the last particle always jumps forward and the wave propagates intact, with no gaps within. This deterministic model is the most compact soliton system in which the wavelength remains constant and equal to the initial length.

Let L_n be the wavelength after n discrete time steps, that is, the total number of particles and gaps in between of the nth configuration. For instance, $L_7 = 10$ in Figure 1. To avoid the trivial situation $L_n \equiv k + n$, we shall always assume

$$p_k > 0. \tag{2}$$

It seems to us that to determine the distribution of L_n, one has to simultaneously solve for the distribution of the positions of all the particles. More precisely, let $D_n^{(j)}$ be the position of the jth particle (that is, the number of particles and gaps within from the jth particle to the wave front (inclusive)). Then $L_n = D_n^{(k)}$, and there is a recursive system of equations among all the

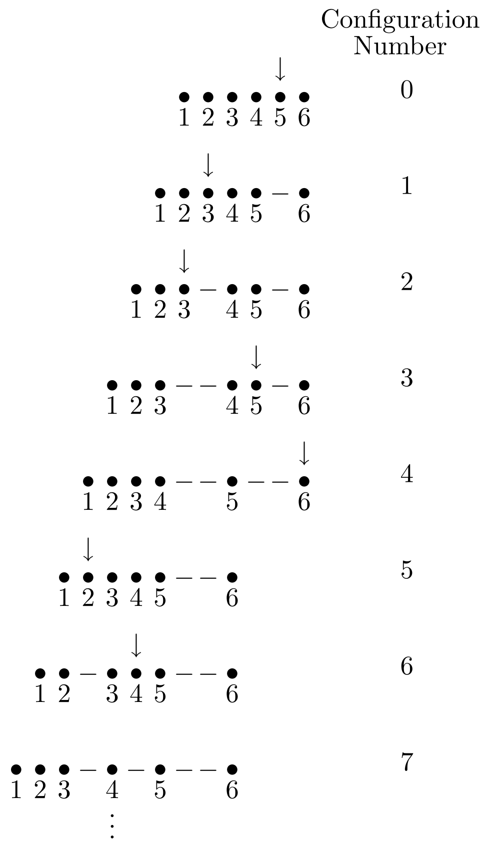

Figure 1: One possible evolution of a soliton wave.

particles' positions. Suppose we are considering the wave evolution from the $(n-1)$ configuration to the nth. We have the row vector $(D^{(1)}_{n-1}, \ldots, D^{(k)}_{n-1})$, and we want to predict the configuration $(D^{(1)}_{n}, \ldots, D^{(k)}_{n})$, after the nth move. If one of the particles $1, \ldots, j-1$ jumps forward in the nth move, the position of the jth particle increases by 1. This happens with collective probability $\sum_{i=1}^{j-1} p_i$. If instead one of the particles $j, \ldots, k$ jumps forward, the particle that was $(j-1)$st in the $(n-1)$st configuration drops to the jth position in the nth configuration. This happens with collective probability $\sum_{i=j}^{k} p_i$. It is evident that the notation

$$q_{a,b} = \sum_{i=a}^{b} p_i \tag{3}$$

will be handy.

Lemma 1

Let $\phi_{j,n}(t) = \mathbf{E}[e^{D_n^{(j)} t}]$ be the moment generating function of the distance $D_n^{(j)}$. Then

$$\phi_{j,n}(t) = e^t \Big(q_{j,k}\phi_{j-1,n-1}(t) + q_{1,j-1}\phi_{j,n-1}(t) \Big). \qquad (4)$$

Proof. For $j = 1, \dots, k$ we have the stochastic recurrence system

$$D_n^{(j)} = \begin{cases} D_{n-1}^{(j-1)} + 1, & \text{with probability } q_{j,k}; \\ D_{n-1}^{(j)} + 1, & \text{with probability } q_{1,j-1}. \end{cases}$$

From the stochastic recurrence we have

$$\mathbf{E}\big[e^{D_n^{(j)} t}\big] = q_{j,k}\mathbf{E}\big[e^{(D_{n-1}^{(j-1)}+1)t}\big] + q_{1,j-1}\mathbf{E}\big[e^{(D_{n-1}^{(j)}+1)t}\big]. \quad \square$$

Remark: The random variables $D_n^{(j)}$ (and consequently L_n) implicitly depend on k, which we hold fixed until asymptotics are later applied on k. We drop k from the notation to avoid triple indexing of the random variables and the corresponding moment generating functions.

3 The steady state

We argue in this section that a soliton propagating according to the proposed model reaches a steady state or a state of equilibrium, for each fixed k. That is, the distribution of the wavelength stabilizes at an eventual *stationary distribution*. We shall show that a limit moment generating function exists for each of $D_n^{(1)}, \dots, D_n^{(k)} = L_n$.

Theorem 1 *For each fixed k there is a moment generating function $\phi_j(t)$ such that $\phi_{j,n}(t) \to \phi_j(t)$, as $n \to \infty$, for each $j = 1, \dots, k$. It follows that $D_n^{(j)}$ converges in distribution to a limiting random variable.*

Proof. We demonstrate by constructive finite induction on j that the sequence $\{\phi_{j,n}(t)\}_{n=1}^{\infty}$ converges. Clearly, $D_n^{(1)} = 1$, that is, $\phi_{1,n}(t) \equiv e^t$, for all $n \ge 1$, establishing a basis for the induction. Assume the existence of a limit $\phi_{j-1}(t)$ for $\phi_{j-1,n}(t)$, as $n \to \infty$. So, for any $\varepsilon > 0$, it is assumed that there exists $n_0 = n_0(\varepsilon, t)$ such that

$$\Big| \phi_{j-1,n}(t) - \phi_{j-1}(t) \Big| < \varepsilon,$$

for all $n \ge n_0$. Iterate the recurrence (4):

$$\begin{aligned} \phi_{j,n}(t) &= q_{1,j-1}^2 e^{2t}\phi_{j,n-2}(t) + q_{j,k}q_{1,j-1}e^{2t}\phi_{j-1,n-2}(t) + q_{j,k}e^t\phi_{j-1,n-1}(t) \\ &\;\;\vdots \\ &= q_{1,j-1}^n e^{nt}\phi_{j,0}(t) + q_{j,k}e^t \sum_{i=0}^{n-1} \big(q_{1,j-1}e^t\big)^{n-i-1}\phi_{j-1,i}(t). \end{aligned}$$

The initial conditions give $\phi_{j,0}(t) = \mathbf{E}[e^{D_0^{(j)}t}] = e^{jt}$. For $n > n_0$, break up the sum as follows:

$$\begin{aligned}\phi_{j,n}(t) &= q_{1,j-1}^n e^{(n+j)t} + q_{j,k}e^t \sum_{i=0}^{n_0-1} \big(q_{1,j-1}e^t\big)^{n-i-1} \phi_{j-1,i}(t) \\ &\quad + q_{j,k}e^t \sum_{i=n_0}^{n-1} \big(q_{1,j-1}e^t\big)^{n-i-1} \phi_{j-1,i}(t).\end{aligned}$$

Take any $t < \ln(1/q_{1,j-1})$. According to the assumption (2) this defines a fixed neighborhood containing 0. Within the range $i \ge n_0$, $\phi_{j-1,i}(t)$ comes close to its limit (within ε distance), and the ϕ terms in the sum up to $n_0 - 1$ are all finite. We have the representation

$$\begin{aligned}\phi_{j,n}(t) &= q_{1,j-1}^n e^{(n+j)t} + O\big(q_{1,j-1}^n e^{nt}\big) \\ &\quad + q_{j,k}e^t \sum_{i=n_0}^{n-1} \big(q_{1,j-1}e^t\big)^{n-i-1} \big(\phi_{j-1}(t) + O(\varepsilon)\big) \\ &= \big(q_{1,j-1}e^t\big)^n e^{jt} + O\big(\big(q_{1,j-1}e^t\big)^n\big) \\ &\quad + q_{j,k}e^t \big(\phi_{j-1}(t) + O(\varepsilon)\big) \sum_{i=n_0}^{n-1} \big(q_{1,j-1}e^t\big)^{n-i-1}.\end{aligned}$$

In the chosen range of t, the term $(q_{1,j-1}e^t)^n$ converges to 0. As $n \to \infty$, the first term is annihilated since $1 \le j \le k$, and k is fixed. The second term also approaches 0; for the same range of t, the remaining sum converges to $1/(1 - q_{1,j-1}e^t)$. It follows that for any $t < \ln(1/q_{1,j-1})$, and for each fixed j, $1 \le j \le k$, a limit exists:

$$\phi_j(t) := \lim_{n\to\infty} \phi_{j,n}(t) = \frac{q_{j,k}e^t \big(\phi_{j-1}(t) + O(\varepsilon)\big)}{1 - q_{1,j-1}e^t}.$$

This expression is valid for any $\varepsilon > 0$. Taking limits as $\varepsilon \to 0$,

$$\phi_j(t) = \frac{q_{j,k}e^t \phi_{j-1}(t)}{1 - q_{1,j-1}e^t}. \tag{5}$$

This completes the induction.

A limit $\phi_j(t)$ for $\phi_{j,n}(t)$ exists, as $n \to \infty$ for any fixed $t < \ln(1/q_{1,j-1})$, and $j = 1, \dots, k$. The limit exists in a fixed neighborhood of 0 that depends only on the given jumping distribution. By Lévy's continuity theorem (see Billingsley (1986)), we have

$$D_n^{(j)} \ \to \ D_j, \qquad \text{for} \ \ n \to \infty,$$

where the convergence mode is in distribution to a limiting random variable D_j (the moment generating function of which is $\phi_j(t)$). □

The length L_n is $D_n^{(k)}$, and therefore has moment generating function $\phi_{k,n}(t) \to \phi_k(t)$. The limit generating function $\phi_k(t)$ is that of a limiting random variable, say $\mathcal{L}_k$. Theorem 1 enables us to find an explicit exact convolution form for the limiting moment generating function.

Theorem 2 *The limiting wavelength $\mathcal{L}_k$ is a convolution of geometric random variables:*

$$\mathcal{L}_k = \text{Geo}(q_{1,k}) + \text{Geo}(q_{2,k}) + \cdots + \text{Geo}(q_{k,k}). \tag{6}$$

Proof. Iterating the recurrence (5) back to $\phi_1(t)$, we have the explicit representation

$$\phi_j(t) = \frac{\prod_{i=2}^{j} q_{i,k}}{\prod_{i=1}^{j-1}(1 - q_{1,i}e^t)}\, e^{jt}.$$

At $j = k$, the latter form can be written as

$$\phi_k(t) = \prod_{i=0}^{k-1} \frac{q_{i+1,k}e^t}{1 - q_{1,i}e^t}. \tag{7}$$

We recognize that the term $1 - q_{1,i}$ is $q_{i+1,k}$ (see the definition of $q_{a,b}$ in (3)). Recalling that the moment generating function of $\text{Geo}(\beta)$ is $\beta e^t/(1-(1-\beta)e^t)$, we see that the limit random variable $\mathcal{L}_k$ is the convolution of the geometric random variables in the statement. □

4 Some normal cases

Our intention in this section and in the sequel is to show that the proposed soliton model is rich enough to exhibit a variety of essentially different limit distributions. In this section we present some normal cases. In the following section we shall see other limits.

We identify conditions on the jumping distribution that lead to limiting normality. Let

$$s_k^2 := \mathbf{Var}[\mathcal{L}_k],$$

and by the convolution (6) it follows that

$$s_k^2 := \sum_{i=1}^{k} \mathbf{Var}\big[\text{Geo}(q_{i,k})\big] = \sum_{i=1}^{k} \frac{q_{1,i-1}}{q_{i,k}^2}.$$

Arbitrarily fix $\varepsilon > 0$. Let

$$R_k(\varepsilon) = \frac{1}{s_k^2} \sum_{j=1}^{k} \int_{\left|x-\frac{1}{q_{j,k}}\right| > \varepsilon s_k} \left| x - \frac{1}{q_{j,k}} \right|^2 dF_{j,k}(x),$$

where $F_{j,k}(x)$ is the distribution function of $\text{Geo}(q_{j,k})$. If this Lindeberg quantity converges to 0, as $k \to \infty$, a central limit theorem will follow. We shall consider cases where $1/q_{j,k} < \lceil \varepsilon s_k \rceil$, and subsequently it is sufficient to check that

$$R_k(\varepsilon) \le R'_k(\varepsilon) = \frac{1}{s_k^2} \sum_{j=1}^{k} \sum_{r=\lceil \varepsilon s_k \rceil}^{\infty} r^2 P\{\text{Geo}(q_{j,k}) = r\} \to 0$$

to prove the limiting normality. From the associated geometric probabilities we have

$$R'_k(\varepsilon) = \frac{1}{s_k^2} \sum_{j=1}^{k} \sum_{r=\lceil \varepsilon s_k \rceil}^{\infty} r^2 q_{j,k} \big(1 - q_{j,k}\big)^{r-1}.$$

We use the identity

$$\sum_{r=a}^{\infty} r^2 x^{r-1} = \frac{(a-1)^2 x^2 + (2a + 1 - 2a^2) x + a^2}{(1-x)^3} x^{a-1}$$

to compute the Lindeberg quantity. If in the resulting expression we separate the terms according to the powers of $\lceil \varepsilon s_k \rceil$, we obtain

$$R'_k(\varepsilon) = \frac{1}{s_k^2} \sum_{j=1}^{k} \Big(A_{j,k}(\varepsilon) \lceil \varepsilon s_k \rceil^2 + B_{j,k}(\varepsilon) \lceil \varepsilon s_k \rceil + C_{j,k}(\varepsilon) \Big),$$

where

$$\begin{aligned} A_{j,k}(\varepsilon) &:= q_{j,k} \big((1 - q_{j,k})^2 - 2(1 - q_{j,k}) + 1 \big) \frac{(1 - q_{j,k})^{\lceil \varepsilon s_k \rceil - 1}}{\big(1 - (1 - q_{j,k})\big)^3} \\ &= {q_{1,j-1}}^{\lceil \varepsilon s_k \rceil - 1}, \end{aligned}$$

$$\begin{aligned} B_{j,k}(\varepsilon) &:= 2 q_{j,k} (1 - q_{j,k}) \big(1 - (1 - q_{j,k})\big) \frac{(1 - q_{j,k})^{\lceil \varepsilon s_k \rceil - 1}}{\big(1 - (1 - q_{j,k})\big)^3} \\ &= 2 \frac{{q_{1,j-1}}^{\lceil \varepsilon s_k \rceil}}{q_{j,k}}, \end{aligned}$$

and

$$C_{j,k}(\varepsilon) = {q_{1,j-1}}^{\lceil \varepsilon s_k \rceil - 1} \frac{2 - q_{j,k}}{q_{j,k}^2}.$$

The tenuous term is $A_k(\varepsilon)$ as it multiplies the highest power of $\lceil \varepsilon s_k \rceil$. If $s_k \to \infty$, as $k \to \infty$, then

$$\frac{1}{s_k^2} \sum_{j=1}^{k} A_{j,k}(\varepsilon) \lceil \varepsilon s_k \rceil^2 = \frac{\lceil \varepsilon s_k \rceil^2}{s_k^2} \sum_{j=1}^{k} A_{j,k}(\varepsilon) = O(\varepsilon^2) \sum_{j=1}^{k} A_{j,k}(\varepsilon).$$

So, if additionally,

$$\sum_{j=1}^{k} A_{j,k}(\varepsilon) \to 0, \tag{8}$$

it follows that

$$\frac{1}{s_k^2}\sum_{j=1}^{k} A_{j,k}(\varepsilon)\lceil \varepsilon s_k\rceil^2 \to 0,$$

for every $\varepsilon > 0$. Once the terms involving $A_{j,k}(\varepsilon)$ converge to 0, so will those involving $B_{j,k}(\varepsilon)$ under mild additional conditions, as can be seen from

$$\begin{aligned}
\frac{1}{s_k^2}\sum_{j=1}^{k} B_{j,k}(\varepsilon)\lceil \varepsilon s_k\rceil &= 2\frac{\lceil \varepsilon s_k\rceil}{s_k^2}\sum_{j=1}^{k}\frac{{q_{1,j-1}}^{\lceil \varepsilon s_k\rceil}}{q_{j,k}} \\
&\le 2\frac{\lceil \varepsilon s_k\rceil}{s_k^2}\sum_{j=1}^{k}\frac{{q_{1,j-1}}^{\lceil \varepsilon s_k\rceil-1}}{q_{k,k}} \\
&= O\Big(\frac{\varepsilon}{p_k s_k}\Big)\sum_{j=1}^{k} A_{j,k}.
\end{aligned}$$

Let us consider solitons with fixed p_k to arrive at an easy condition, so that under the condition (8) the last term goes to 0. Lastly,

$$\begin{aligned}
\frac{1}{s_k^2}\sum_{j=1}^{k} C_{j,k}(\varepsilon) &\le \frac{2}{s_k^2}\sum_{j=1}^{k}\frac{{q_{1,j-1}}^{\lceil \varepsilon s_k\rceil-1}}{q_{j,k}^2} \\
&\le \frac{2}{p_k^2 s_k^2}\sum_{j=1}^{k} A_{j,k},
\end{aligned}$$

which by (8) and the assumption that p_k is a constant also approaches 0, as $k \to \infty$. Thus, for every fixed $\varepsilon > 0$, all three parts of $R'_k(\varepsilon)$ converge to 0, as $k \to \infty$.

To summarize, one has the following Gaussian tendency

$$\frac{\mathcal{L}_k - \mathbf{E}[\mathcal{L}_k]}{\sqrt{\mathbf{Var}[\mathcal{L}_k]}} \xrightarrow{\mathcal{D}} \mathcal{N}(0,1),$$

if $s_k \to \infty$, p_k is a constant, and (8) holds.

As an illustrating example, consider a soliton of k particles always labeled $1, \ldots, k$ from left to right, with the jumping distribution

$$p_j = \frac{1}{2(k-1)}, \qquad \text{for} \ \ k = 1, \ldots, k-1, \qquad\qquad p_k = \frac{1}{2}.$$

We shall demonstrate that the stationary distribution of the (normalized) wavelength of this soliton is normal, when $k \to \infty$, as in real physical systems. For $j = 1, \dots, k$, we have $q_{j,k} = \frac{2k-j-1}{2(k-1)}$. And so,

$$\begin{aligned}\mathbf{E}[\mathcal{L}_k] &= \sum_{j=1}^{k} \frac{1}{q_{j,k}} \\ &= \sum_{j=1}^{k} \frac{2(k-1)}{2k-j-1}\end{aligned}$$

and

$$\begin{aligned}\mathbf{Var}[\mathcal{L}_k] &= s_k^2 \\ &= \sum_{j=1}^{k} \frac{1-q_{j,k}}{q_{j,k}^2} \\ &= 2(k-1)\sum_{j=2}^{k} \frac{j-1}{(2k-j-1)^2}\end{aligned}$$

which, after the change of summation variable $j = 2k - r - 1$, transform into relations expressible in terms of harmonic numbers:

$$\mathbf{E}[\mathcal{L}_k] = 2(k-1)\big(H_{2k-2}^{(1)} - H_{k-2}^{(1)}\big),$$

and

$$\begin{aligned}\mathbf{Var}[\mathcal{L}_k] &= 2(k-1)\sum_{r=k-1}^{2k-3} \frac{2k-r-2}{r^2} \\ &= 4(k-1)^2\big(H_{2k-3}^{(2)} - H_{k-2}^{(2)}\big) - 2(k-1)\big(H_{2k-3}^{(1)} - H_{k-2}^{(1)}\big).\end{aligned}$$

Using the standard approximation for the harmonic numbers, we have the asymptotic equivalents

$$\begin{aligned}\mathbf{E}[\mathcal{L}_k] &\sim (2\ln 2)k, \\ \mathbf{Var}[\mathcal{L}_k] &\sim 2(1-\ln 2)k.\end{aligned}$$

Then, $s_k \to \infty$, when $k \to \infty$. Subsequently, the condition (8) checks out as follows:

$$\begin{aligned}\sum_{j=1}^{k} A_{j,k}(\varepsilon) &= \sum_{j=1}^{k} \Big(\frac{j-1}{2(k-1)}\Big)^{\lceil \varepsilon s_k \rceil - 1} \\ &= O\Big(\frac{k}{2^{\lceil \varepsilon s_k \rceil}\lceil \varepsilon s_k \rceil}\Big).\end{aligned}$$

It follows that the steady state wavelength approaches a Gaussian law:

$$\frac{\mathcal{L}_k - (2\ln 2)k}{\sqrt{k}} \xrightarrow{\mathcal{D}} \mathcal{N}\big(0, 2(1-\ln 2)\big).$$

5 Aging solitons

We consider in this section an application of the general theory presented in Section 3. Let us investigate an exponential aging system in which particles that did not jump for a while are given higher jumping probability according to the jumping distribution (1).

5.1 The degenerate uniform case

The degenerate case $r = 1$ presents a uniform aging system, all particles in a configuration (except the wave front) are equally likely to jump next, and so $p_2 = p_3 = \cdots = p_k = 1/(k-1)$, and $p_1 = 0$. In this uniform jumping distribution we have

$$q_{i,k} = p_i + \cdots + p_k = \frac{k-i+1}{k-1}, \qquad \text{for} \quad i \geq 2,$$

and $q_{1,k} = 1$. Taking expectations of (6), we get an exact average for the stationary distribution:

$$\begin{aligned}
\mathbf{E}[\mathcal{L}_k] &= 1 + \sum_{i=2}^{k} \frac{1}{q_{i,k}} \\
&= 1 + \sum_{i=2}^{k} \frac{k-1}{k-i+1} \\
&= (k-1)H_{k-1}^{(1)} + 1.
\end{aligned}$$

By a similar calculation we get an exact variance; take the variance of (6) and use the independence of the geometric random variables in the convolution to get

$$\begin{aligned}
\mathbf{Var}[\mathcal{L}_k] &= \sum_{i=2}^{k} \frac{1-q_{i,k}}{q_{i,k}^2} \\
&= (k-1)\sum_{i=1}^{k-1} \frac{i-2}{(k-i+1)^2} \\
&= (k-1)\sum_{i=1}^{k-1} \frac{(k-3)-(k-i-1)}{(k-i+1)^2} \\
&= (k-1)(k-3)H_{k-1}^{(2)} - (k-1)H_{k-1}^{(1)}.
\end{aligned}$$

In a physical system made up of particles, k will be enormously large. As $k \to \infty$, we have the asymptotic equivalents

$$\begin{aligned} \mathbf{E}[\mathcal{L}_k] &\sim k \ln k, \\ \mathbf{Var}[\mathcal{L}_k] &\sim \frac{\pi^2}{6} k^2. \end{aligned}$$

The relatively small variance suggests sharp concentration under the appropriate scaling. Indeed, by Chebyshev's inequality (for any fixed ε) we have

$$P\Big\{\Big|\frac{\mathcal{L}_k}{k \ln k} - 1\Big| < \varepsilon\Big\} \le \frac{\mathbf{Var}[\mathcal{L}_k]}{\varepsilon^2 k^2 \ln^2 k} \sim \frac{\pi^2}{6\varepsilon \ln^2 k} \to 0.$$

So,

$$\frac{\mathcal{L}_k}{k \ln k} \to 1, \qquad \text{in probability.}$$

Properly centered and scaled, the length of a uniform soliton converges in distribution. One can find this out from the exact moment generating function (7), which assumes the form

$$\begin{aligned} \phi_k(t) &= e^{kt} \prod_{j=2}^{k-1} \frac{(k-j)/(k-1)}{1-(j-1)e^t/(k-1)} \\ &= \frac{(k-2)!\, e^{kt}}{e^{(k-2)t}((k-1)e^{-t}-1)((k-1)e^{-t}-2)\ldots((k-1)e^{-t}-(k-2))} \\ &= \frac{\Gamma\Big((k-1)e^{-t}-(k-2)\Big)\,\Gamma(k-1)}{\Gamma\Big((k-1)e^{-t}\Big)} e^{2t} \\ &= \frac{\Gamma\Big(1-(k-1)t+O(kt^2)\Big)\,\Gamma(k-1)}{\Gamma\Big((k-1)-(k-1)t+O(kt^2)\Big)} e^{2t}. \end{aligned}$$

Set $t = u/k$. Stirling's approximation to the gamma function,

$$\frac{\Gamma(z+a)}{\Gamma(z+b)} \sim z^{a-b}, \qquad \text{as} \ \ z \to \infty,$$

can be used here. When k is large, we have

$$\phi_k\Big(\frac{u}{k}\Big) = k^{u+O(1/k)}\,\Gamma\Big(1-u\Big(\frac{1}{k}\Big)\Big),$$

or

$$\phi_k\Big(\frac{u}{k}\Big)e^{-u\ln k} \to \Gamma(1-u),$$

the right hand-side being the moment generating function of Gumbel's distribution (a special case of the class of Fisher-Trippett extreme-value distributions), with distribution function $e^{-e^{-x}}$. That is,

$$\frac{\mathcal{L}_k - k\ln k}{k} \xrightarrow{\mathcal{D}} Y,$$

where the limiting random variable Y has Gumbel's distribution.

5.2 The nonuniform case

Consider a bona fide nondegenerate aging soliton as in (1), with $r > 1$. The exact moment generating function for this aging soliton is:

$$\phi_k(t) = e^{kt} \prod_{i=1}^{k-1} \frac{r^i + \cdots + r^{k-1}}{(r + \cdots + r^{k-1}) - (r + \cdots + r^{i-1})e^t}.$$

Summing each geometric series we arrive at the representation

$$\phi_k(t) = e^{kt} \prod_{i=1}^{k-1} \frac{r^{k-1} - r^{i-1}}{(r^{k-1} - 1) - (r^{i-1} - 1)e^t}.$$

For instance, with $r = 2$,

$$\phi_5(t) = \frac{1344e^{5t}}{(15 - e^t)(15 - 3e^t)(15 - 7e^t)},$$

which is the moment generating function of 2 plus three geometric random variables. For any finite k, $\phi_k(t) \to e^{kt}$, as $r \to \infty$, recovering the nature of one of the deterministic waves in the cellular automaton model of Takahashi and Satsuma (1990) for colliding solitons. By local expansion we have

$$\begin{aligned} \phi_k(t)e^{-kt} &= 1 + \theta_1 t + \theta_2 \frac{t^2}{2} + \theta_3 \frac{t^3}{6} + \cdots \\ &:= g_{X_r}(t), \end{aligned}$$

where $g_{X_r}(t)$ is the moment generating function of some random variable X_r, and θ_i are constants. In other words,

$$\mathcal{L}_k - k \xrightarrow{\mathcal{D}} X_r.$$

Note that only centering is needed, but no scaling ($\mathbf{Var}[\mathcal{L}_k]$ remains finite, as $k \to \infty$). For instance, when $r = 2$, the limit moment generating function has the expansion

$$g_{X_2}(t) = 1 + 6.196428t + 20.212691t^2 + 46.785693t^3 + 87.44099t^4 + \cdots ,$$

with all coefficients approximated by their correct first six decimal places.

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