Two-dimensional Burgers Cellular Automaton

Katsuhiro Nishinari, Junta Matsukidaira

Department of Applied Mathematics and Informatics,

Ryukoku University, Seta, Ohtsu 520-2194, JAPAN

and

Daisuke Takahashi

Department of Mathematical Sciences,

Waseda University, Ohkubo, Shinjuku-ku, Tokyo 169-8555, JAPAN

Abstract

In this paper, a two-dimensional cellular automaton(CA) associated with a two-dimensional Burgers equation is presented. The 2D Burgers equation is an integrable generalization of the well-known Burgers equation, and is transformed into a 2D diffusion equation by the Cole-Hopf transformation. The CA is derived from the 2D Burgers equation by using the ultradiscrete method, which can transform dependent variables into discrete ones. Some exact solutions of the CA, such as shock wave solutions, are studied in detail.

1 Introduction

Recently, cellular automaton(CA) is extensively studied as a method to make models of complex systems, such as traffic flow[1], fluid dynamics[2], immune systems[3]. Since CA models are represented by binary procedures or discrete equations, exact analysis or explicit solutions are rarely obtained. Thus we often use numerical calculations or thermodynamical and statistical methods[4][5] to study global features.

Recently, the ultradiscrete method is proposed as a new type of discretization technique which allows us to study direct relations between continuous system and CA[6][7]. As for soliton equations, we can derive associated CA from a continuous equation keeping their integrable properties by the method. Another example of applying this method has been given by two of the authors and a CA associated with the 1D Burgers equation is derived as follows[8]:

$$U_i^{t+1} = U_i^t + \min(U_{i-1}^t, L - U_i^t) - \min(U_i^t, L - U_{i+1}^t). \tag{1}$$

The CA is called Burgers CA(BCA), and it has nice properties; it can be transformed into a linear CA of diffusion type, and is a multi-value generalization of the rule–184 CA[9]. Moreover, BCA and its extensions can be used as highway traffic models, and properties of models and observed data are compared in detail[10][11].

In this paper, we generalize the previous results to two spatial dimension. First, we introduce 2D Burgers equation, which can be linearized by a dependent variable transformation. Then we discretize independent variables keeping its integrable properties. Finally, we use the ultradiscrete method to obtain two-dimensional BCA. We also study some exact solutions of the CA.

2 Two-dimensional Burgers equation and its ultradiscretization

2D Burgers equation we consider in this paper is

$$u_t = u_{xx} + u_{yy} + 2uu_x + 2vu_y, \tag{2}$$

$$v_x = u_y, \tag{3}$$

or, eliminating the variable v, we can rewrite these into a single equation

$$u_t = u_{xx} + u_{yy} + 2uu_x + 2u_y \int u_y dx. \tag{4}$$

This equation has a remarkable property that it can be linearized by the Cole–Hopf transformation

$$u = \frac{f_x}{f} \tag{5}$$

into a 2D diffusion equation

$$f_t = f_{xx} + f_{yy}. \tag{6}$$

Thus the Burgers equations are integrable in a sense that they can be transformed into a linear equation. It is noted that from (3) and (5), we obtain

$$v = \frac{f_y}{f}. \tag{7}$$

From (5) and (7) it is apparent that the variables u and v can be treated symmetrically, then we can derive the time evolution equation for v. From (6) and (7), we obtain

$$v_t = v_{xx} + v_{yy} + 2vv_y + 2uv_x. \tag{8}$$

Thus we can write 2D Burgers equation in another coupled form as

$$\mathbf{u}_t = \nabla^2 \mathbf{u} + 2(\mathbf{u} \cdot \nabla)\mathbf{u}, \tag{9}$$

where $\mathbf{u} = (u, v)$. This form appears in [12], in which it is called vector Burgers equation.

To discretize the independent variables in (2) and (3), we utilize discrete analogues to (5) and (6)[8]. Discretizing both time and space variables in (6), we obtain a discrete diffusion equation

$$f_{ij}^{t+1} = (1-4\delta)f_{ij}^t + \delta(f_{i+1j}^t + f_{i-1j}^t + f_{ij+1}^t + f_{ij-1}^t) \tag{10}$$

where $\delta = \Delta t/h^2$ and $h = \Delta x = \Delta y$. $\Delta t, \Delta x$ and Δy are lattice intervals in time t, space x and y, respectively. Next we define discrete analogues to the Cole–Hopf transformation (5) and (7)

$$u_{ij}^t \equiv c\frac{f_{i+1j}^t}{f_{ij}^t}, \quad v_{ij}^t \equiv c\frac{f_{ij+1}^t}{f_{ij}^t} \tag{11}$$

where c is a constant. Evolution equations for u_{ij}^t and v_{ij}^t are derived from (10) and (11) as

$$u_{ij}^{t+1} = u_{ij}^t \frac{\dfrac{1-4\delta}{c\delta} + \dfrac{u_{i+1j}^t}{c^2} + \dfrac{1}{u_{ij}^t} + \dfrac{v_{i+1j}^t}{c^2} + \dfrac{1}{v_{i+1j-1}^t}}{\dfrac{1-4\delta}{c\delta} + \dfrac{u_{ij}^t}{c^2} + \dfrac{1}{u_{i-1j}^t} + \dfrac{v_{ij}^t}{c^2} + \dfrac{1}{v_{ij-1}^t}}, \tag{12}$$

$$\frac{v_{i+1j}^t}{v_{ij}^t} = \frac{u_{ij+1}^t}{u_{ij}^t}. \tag{13}$$

If we take an appropriate continuous limit $\Delta t, h \to 0$, we obtain (2) and (3) from (12) and (13). Moreover, (12) and (13) can be transformed into the linear diffusion equation (10) through (11).

Next, we ultradiscretize (12) and (13), that is, discretize dependent variables u and v. Let us introduce a transformation of variables and parameters as follows:

$$u_{ij}^t = e^{U_{ij}^t/\varepsilon}, \quad v_{ij}^t = e^{V_{ij}^t/\varepsilon}, \tag{14}$$

$$\frac{1-4\delta}{c\delta} = e^{-M/\varepsilon}, \tag{15}$$

$$c^2 = e^{L/\varepsilon}, \tag{16}$$

where ε is a parameter. Substituting these transformations into (12) and (13), taking a limit $\varepsilon \to +0$ and using the formula

$$\lim_{\varepsilon\to+0} \varepsilon \log(\exp(\frac{A}{\varepsilon}) + \exp(\frac{B}{\varepsilon})) = \max(A, B) = -\min(-A, -B), \qquad (17)$$

we obtain

$$\begin{aligned} U_{ij}^{t+1} &= U_{ij}^t + \min(M, U_{i-1j}^t, L - U_{ij}^t, V_{ij-1}^t, L - V_{ij}^t) \\ &\quad - \min(M, U_{ij}^t, L - U_{i+1j}^t, V_{i+1j-1}^t, L - V_{i+1j}^t), \qquad (18) \\ V_{i+1j}^t - V_{ij}^t &= U_{ij+1}^t - U_{ij}^t. \qquad (19) \end{aligned}$$

These are two-dimensional generalization of BCA (1). We call this system two-dimensional Burgers CA(2DBCA). In (18), we can eliminate the variable V by using (19) and obtain

$$U_{ij}^{t+1} = U_{ij}^t + Q_{ij}^t - Q_{i+1j}^t, \qquad (20)$$

where flow Q_{ij}^t is given by

$$Q_{ij}^t = \min(M, U_{i-1j}^t, L - U_{ij}^t, V_{0j-1}^t + \sum_{k=0}^{i-1}(U_{kj}^t - U_{kj-1}^t), L - V_{0j}^t - \sum_{k=0}^{i-1}(U_{kj+1}^t - U_{kj}^t)), \quad (21)$$

which is an ultradiscrete version of (4). In (21), V_{0j}^t represents boundary conditions for V at $i = 0$. Since variables u and v can be treated symmetrically, we can derive an equation for V by using a similar procedure to the above. The result is

$$\begin{aligned} V_{ij}^{t+1} = V_{ij}^t &+ \min(M, U_{i-1j}^t,\, L - U_{ij}^t,\, V_{ij-1}^t,\, L - V_{ij}^t) \\ &- \min(M, U_{i-1j+1}^t,\, L - U_{ij+1}^t,\, V_{ij}^t,\, L - V_{ij+1}^t). \qquad (22) \end{aligned}$$

It should be noted that 2DBCA is also expressed by a set of equations (18) and (22), and it is equivalent to that of (18) and (19) under an appropriate condition. This fact is directly shown as follows. Let us define h_{ij}^t by

$$h_{ij}^t \equiv \min(M, U_{ij}^t,\, L - U_{i+1j}^t,\, V_{i+1j-1}^t,\, L - V_{i+1j}^t), \qquad (23)$$

then (18) and (22) are written as

$$\begin{aligned} U_{ij}^{t+1} &= U_{ij}^{t} + h_{i-1j}^{t} - h_{ij}^{t} \qquad (24) \\ V_{ij}^{t+1} &= V_{ij}^{t} + h_{i-1j}^{t} - h_{i-1j+1}^{t}. \qquad (25) \end{aligned}$$

By using these equation we have

$$\begin{aligned} & V_{i+1j}^{t+1} - V_{ij}^{t+1} - U_{ij+1}^{t+1} + U_{ij}^{t+1} \\ = \; & \{V_{i+1j}^{t} + h_{ij}^{t} - h_{ij+1}^{t}\} - \{V_{ij}^{t} + h_{i-1j}^{t} - h_{i-1j+1}^{t}\} \\ & -\{U_{ij+1}^{t} + h_{i-1j+1}^{t} - h_{ij+1}^{t}\} + \{U_{ij}^{t} + h_{i-1j}^{t} - h_{ij}^{t}\} \\ = \; & V_{i+1j}^{t} - V_{ij}^{t} - U_{ij+1}^{t} + U_{ij}^{t}. \qquad (26) \end{aligned}$$

Thus

$$V_{i+1j}^{t} - V_{ij}^{t} - U_{ij+1}^{t} + U_{ij}^{t} = \text{const.} \qquad (27)$$

holds for any time t. By choosing initial condition of U and V so that the above constant becomes zero everywhere, then (19) is always satisfied. Note that if U is periodic in lattice i and V in j, then the total sums $\sum_i U_{ij}^t$ and $\sum_j V_{ij}^t$ are conserved quantities, which is directly shown by using (18) and (22).

Finally, we consider a relation between the ultradiscrete Burgers equation and ultradiscrete diffusion equation. Introducing a transformation

$$f_{ij}^{t} = \exp(F_{ij}^{t}/\varepsilon), \qquad (28)$$

ultradiscrete Cole–Hopf transformations

$$\begin{aligned} U_{ij}^{t} &= F_{i+1j}^{t} - F_{ij}^{t} + \frac{L}{2}, \qquad (29) \\ V_{ij}^{t} &= F_{ij+1}^{t} - F_{ij}^{t} + \frac{L}{2} \qquad (30) \end{aligned}$$

are obtained from (11) and (14) under the limit $\varepsilon \to +0$. Substituting (28) into (10), we obtain an ultra-discrete diffusion equation;

$$F_{ij}^{t+1} = \max(F_{ij}^{t} + \frac{L}{2} - M, F_{i+1j}^{t}, F_{i-1j}^{t}, F_{ij+1}^{t}, F_{ij-1}^{t}) - \max(0, \frac{L}{2} - M). \qquad (31)$$

This is also obtained as follows. If we substitute (29) and (30) into (18), we have

$$\begin{aligned} F_{i+1j}^{t+1} - F_{ij}^{t+1} &= \max(F_{ij}^t + \frac{L}{2} - M, F_{i+1j}^t, F_{i-1j}^t, F_{ij+1}^t, F_{ij-1}^t) \\ &- \max(F_{i+1j}^t + \frac{L}{2} - M, F_{i+2j}^t, F_{ij}^t, F_{i+1j+1}^t, F_{i+1j-1}^t). \end{aligned} \tag{32}$$

Since this equation is in a recurrence form in i, we obtain (31) if we choose the decoupling constant in the above equation as $-\max(0, L/2 - M)$. This shows that we can linearize 2DBCA by the ultradiscrete Cole-Hopf transformations and the CA can be considered to keep integrable property. (19) is automatically satisfied by substituting (29) and (30) into it, and this means that (19) represents the compatibility condition of differences in i and j direction of the variable F. Note that we consider a linearity of (31) in a sense that superposition of solutions is realized by max operation like '+' in (6).

3 Some solutions of 2DBCA

3.1 Shock wave solution

In this section, we discuss some solutions and examine their behavior in detail. First, we derive an exact solution which represents shock waves. In order to obtain the solution, we utilize the shock wave solution of the discrete Burgers equation (12) and (13). Let us assume f_{ij}^t has the following form

$$f_{ij}^t = 1 + \sum_{z=1}^{N} \exp(k_z i + r_z j + \omega_z t + \xi_z), \tag{33}$$

where k_z, r_z, ω_z and ξ_z $(z = 1, 2, \cdots, N)$ are constants. (33) is an exact solution of (10) under the condition

$$\omega_z = \log(1 - 4\delta + \delta(e^{k_z} + e^{-k_z} + e^{r_z} + e^{-r_z})) \quad (z = 1, 2, \cdots, N), \tag{34}$$

which is obtained simply substituting (33) into (10). (34) is the dispersion relation between the frequency (ω_z) and the wavenumber (k_z and r_z). From (11) we have

$$u_{ij}^t = c\frac{1+\sum_{z=1}^N \exp(k_z(i+1)+r_zj+\omega_zt+\xi_z)}{1+\sum_{z=1}^N \exp(k_zi+r_zj+\omega_zt+\xi_z)}, \tag{35}$$

and also have v_{ij}^t. We call this "N-shock wave" solution of (12) and (13). From this solution we can obtain a shock wave solution of 2DBCA by the ultradiscrete method. First we assume

$$k_z = \frac{K_z}{\varepsilon},\ r_z = \frac{R_z}{\varepsilon},\ \omega_z = \frac{\Omega_z}{\varepsilon},\ \xi_z = \frac{\Xi_z}{\varepsilon}, \tag{36}$$

and noticing (14) and (16), we obtain from (35)

$$\begin{aligned} U_{ij}^t &= \frac{L}{2} + \max(0, \theta_1(i+1,j,t), \theta_2(i+1,j,t), \cdots, \theta_N(i+1,j,t)) \\ &- \max(0, \theta_1(i,j,t), \theta_2(i,j,t), \cdots, \theta_N(i,j,t)), \end{aligned} \tag{37}$$

by taking the limit $\varepsilon \to +0$. Here phases $\theta_z(z=1,\cdots,N)$, are given by

$$\theta_z(i,j,t) = K_zi + R_zj + \Omega_zt + \Xi_z. \tag{38}$$

From (34), a dispersion relation becomes in the limit

$$\Omega_z = \max(|K_z|, |R_z|, \frac{L}{2} - M) - \max(0, \frac{L}{2} - M). \tag{39}$$

(37) is an exact solution of (18) and we also get a solution V_{ij}^t by (19), which represent an ultradiscrete shock wave. The max operation in (37) represents the superposition principle, as mentioned in the previous section.

Fig.1 shows an example of shock wave solution (37) in the case of $N=2$. Parameters are taken as $K_1=-1, R_1=K_2=R_2=1$, $\Xi_1=\Xi_2=0$, $L=2$ and $M=2$. We easily see that $\lim_{i\to\infty} U_{ij}^t = L/2+K_2=2$, and $\lim_{i\to-\infty} U_{ij}^t = L/2+K_1=0$, and $\lim_{j\to-\infty} U_{ij}^t = L/2=1$.

3.2 Particle model and excitation behavior

If we assume the parameter M is sufficiently larger than $L/2$, we can neglect terms that contain M in (18), (22) and (31), and simplify the analysis on them. In this case equations on U and V become

$$\begin{aligned} U_{ij}^{t+1} &= U_{ij}^t + \min(U_{i-1j}^t, L - U_{ij}^t, V_{ij-1}^t, L - V_{ij}^t) \\ &\quad - \min(U_{ij}^t, L - U_{i+1j}^t, V_{i+1j-1}^t, L - V_{i+1j}^t), \qquad (40) \\ V_{ij}^{t+1} &= V_{ij}^t + \min(U_{i-1j}^t,\ L - U_{ij}^t,\ V_{ij-1},\ L - V_{ij}^t) \\ &\quad - \min(U_{i-1j+1}^t,\ L - U_{ij+1}^t,\ V_{ij},\ L - V_{ij+1}^t), \qquad (41) \end{aligned}$$

and equation on F becomes

$$F_{ij}^{t+1} = \max(F_{i+1j}^t, F_{i-1j}^t, F_{ij+1}^t, F_{ij-1}^t). \qquad (42)$$

It is easily verified that if initial conditions are taken as $0 \le F_{ij}^0, 0 \le U_{ij}^0 \le L$ and $0 \le V_{ij}^0 \le L$, then these conditions hold for any time t.

The set of equations (40) and (41) express a simple particle model. Before explaining the model, we shortly review a one-dimensional particle model defined by (1) described in [8]. In the model, there are one-dimensional infinite sites indexed by i. Every site can hold L particles at most. Particles at each site move to empty spaces in its neighboring right site synchronously at each time step. According to this rule, flow from site i to $i+1$ at time t becomes $\min(U_i^t, L - U_{i+1}^t)$ if U_i^t denotes the number of particles at site i and time t. Thus considering the number of particles coming into and escaping from site i, we obtain (1).

Next we explain two-dimensional particle model expressed by (40) and (41). The model is constructed as follows:

(a) There are two-dimensional infinite sites indexed by i and j.

(b) There are two kinds of particles, shortly saying U-particles and V-particles. U-particles move only in positive i direction. V-particles in positive j direction, as shown in Fig.2.

(c) Every site can hold L particles at most of both kinds respectively.

(d) U-particles at site (i,j) can move to empty spaces in its neighboring right site $(i+1,j)$ per unit time. V-particles at (i,j) can move to those in up site $(i,j+1)$.

(e) The number of U-particles from $(i-1,j)$ to (i,j) from t to $t+1$ must be equal to that of V-particles from $(i,j-1)$ to (i,j) at the same time step.

(f) Under this restriction, U- and V-particles move to occupy neighboring empty spaces.

Let us define the number of U- and V-particles at site (i,j) and time t by U_{ij}^t and V_{ij}^t respectively. Then, according to the above rule (e) and (f), flow of U-particles from $(i-1,j)$ to (i,j) and that of V-particles from $(i,j-1)$ to (i,j) are both $\min(U_{i-1j}^t, L-U_{ij}^t, V_{ij-1}^t, L-V_{ij}^t)$. Thus, evolution equations of the above model become (40) and (41). We call this rule "pairing rule" because the same number of U- and V-particles come into the same site.

We show simple examples of time evolution of the above model. We choose $L=2$ for all examples below. As the first example, we set $F_{ij}^0=0$ except $F_{0,0}^0=1$ at the origin. This means $U_{-1,0}^0=2$, $U_{0,0}^0=0$, $V_{0,-1}^0=2$, $V_{0,0}^0=0$ and $U_{ij}^0=V_{ij}^0=1$ at all other points. The time evolution is given in Fig.3. An exact solution of (42) is

$$F_{ij}^t = \max_{\substack{|k+l|\le t \\ |k-l|\le t \\ k+l=t \bmod 2}} G_{i+i_0-k,\,j+j_0-l}, \tag{43}$$

where $G_{k,l}$ is defined by

$$\begin{aligned} G_{k,l} \;=\; & \min(\max(0,2k+1)-2\max(0,2k)+\max(0,2k-1), \\ & \max(0,2l+1)-2\max(0,2l)+\max(0,2l-1)), \end{aligned} \tag{44}$$

and i_0, j_0 is real phase constants. Since i_0 and j_0 are arbitrary real constants, the above solution satisfies (42) even if i–j lattice is shifted as $i \to i + i'_0$ and $j \to j + j'_0$. If we set $i_0 = j_0 = 0$ in (43), we obtain the same solution shown in Fig. 3 on integer lattice points. We can derive the above solution from a solution of difference equation (10) using ultradiscretization. The derivation is shown in Appendix. The solution represents an expansion of excited wave of F as shown in Fig.3(a). The solutions for U and V are obtained by substituting (43) into (29) and (30), respectively, or simply by moving U-particles and V-particles according to the above rule. Fig.4 is an example of expanding rings of U and V. Initial conditions are $F^0_{i,j} = 0$ except $F^0_{0,0} = 1$ and $F^0_{1,0} = 1$. The solution is apparently obtained by a superposition of (43). We see that $\sum_i U^t_{ij}$ and $\sum_j V^t_{ij}$ are constant for time in both cases.

4 Concluding discussions

In this paper, two-dimensional CA associated with Burgers equation(2DBCA) is presented by using ultradiscrete method. 2DBCA has some remarkable properties that it can be linearized by means of the ultradiscrete Cole–Hopf transformation, and has exact solutions which shows N-shock wave and excitation behaviors.

We also show a particle model of the CA, which can be considered as a pairing movement of particles in i and j direction. We hope that this rule and its integrability contribute to make models on some multi-dimensional complex systems.

Acknowledgment

The authors are grateful to Prof. Ryogo Hirota and Prof. Yasuhiro Ohta for fruitful discussions and helpful comments. This work is partially supported by Grant-in-Aid from the Ministry of Education, Science and Culture.

References

[1] K. Nagel and M. Schreckenberg, J. Phys. I France **2** (1992) p.2221.

[2] G. McNamara and G. Zaneti, Phys. Rev. Lett. **61** (1988) p.2332.

[3] D. Chowdhury and D. Stauffer, J. Stat. Phys. **59** (1990) p.1019.

[4] M. Schreckenberg, A. Schadschneider, K. Nagel and N. Ito, Phys. Rev. E **51** (1995) p.2939.

[5] B. Derrida, M. R. Evans, V. Hakim and V. Pasquier, J. Phys. A **26** (1993) p.1493.

[6] T. Tokihiro, D. Takahashi, J. Matsukidaira and J. Satsuma, Phys. Rev. Lett. **76** (1996) p.3247.

[7] J. Matsukidaira, J. Satsuma, D. Takahashi, T. Tokihiro and M. Torii, Phys. Lett. A **225** (1997) p.287.

[8] K. Nishinari and D. Takahashi, J. Phys. A **31** (1998) p.5439.

[9] S. Wolfram, *Theory and Applications of Cellular Automata* (World Scientific, Singapore, 1986).

[10] K. Nishinari and D. Takahashi, J. Phys. A **32** (1999) p.93.

[11] K. Nishinari and D. Takahashi, to appear in J. Phys. A.

[12] K. B. Wolf, L. Hlavaty and S. Steinberg, J. Math. Anal. and Applic. **114** (1986) p.340.

Appendix

In this appendix, we show that (43) can be derived from a solution of the difference equation (10) through the ultradiscretization.

First, since we consider the case that the parameter M is sufficiently large, we can set $\delta = 1/4$ by using (15). Then (10) becomes

$$f_{ij}^{t+1} = \frac{1}{4}(f_{i+1j}^t + f_{i-1j}^t + f_{ij+1}^t + f_{ij-1}^t). \tag{45}$$

We can interpret this equation as 2D random walk process. Therefore, we obtain an exact solution including 2D binomial distribution,

$$f_{ij}^t = \sum_{\substack{|k+l|\le t \\ |k-l|\le t \\ k+l=t \mod 2}} c_{k,l}^t \, g_{i+i_0-k,\, j+j_0-l}, \tag{46}$$

where

$$\begin{aligned} c_{k,l}^t &= \frac{1}{4^t}\cdot\frac{t!}{((t-|k+l|)/2)!((t+|k+l|)/2)!}\cdot\frac{t!}{((t-|k-l|)/2)!((t+|k-l|)/2)!} \\ g_{k,l} &= 1\Big/\Big\{\frac{(1+\exp(2k/\varepsilon))^2}{(1+\exp((2k+1)/\varepsilon))(1+\exp((2k-1)/\varepsilon))} \\ &\qquad +\frac{(1+\exp(2l/\varepsilon))^2}{(1+\exp((2l+1)/\varepsilon))(1+\exp((2l-1)/\varepsilon))}\Big\} \end{aligned}$$

and i_0, j_0 is real phase constants, ε is a limiting parameter. We have

$$\lim_{\varepsilon\to+0} \varepsilon \log g_{k,l} = G_{k,l} \text{ in } (44)$$

and $c_{k,l}^t$ is always positive finite for finite k, l and t. Therefore, taking a limit $\lim_{\varepsilon\to+0} \varepsilon\log f_{ij}^t$, all $c_{k,l}^t$ terms disappear and only $g_{k,l}$ survive in the summation of (46). Thus we obtain F_{ij}^t in (43) from f_{ij}^t in (46) by the ultradiscretization. Figure A1 (a) shows F_{ij}^t , and (b) and (c) show $\varepsilon \log f_{ij}^t$. We draw all figures as continuous plot on i and j and set $t = 2$ and $i_0 = j_0 = 0$. Parameter ε is 0.2 and 0.01 in (b) and (c), respectively. From these figures, we can see a difference solution converges to a ultradiscrete one as $\varepsilon \to +0$.

Figure Captions

Fig.1 The snapshot of shock wave solution at $t = 1$ in the case of $N = 2$. Parameters are $K_1 = -1, R_1 = K_2 = R_2 = 1$, $\Xi_1 = \Xi_2 = 0$, $L = 2$ and $M = 2$.

Fig.2 A physical explanation of the (2+1)-dimensional BCA, called "pairing rule". The same number of U-particles and V-particles come into the same site. The solid arrow and broken arrow show the movement of U and V, respectively.

Fig.3 Excitation behavior of (a)F, (b)U, (c)V at $t = 6$. We choose $L = 2$, and initial conditions are $F^0_{i,j} = 0$ except the origin $F^0_{0,0} = 1$. The light gray, dark gray and black square represent 0, 1, 2, respectively.

Fig.4 Expanding wave of U and V ((a)F, (b)U, (c)V at $t = 6$). Initial conditions are $F^0_{i,j} = 0$ except $F^0_{0,0} = 1$ and $F^0_{1,0} = 1$. The light gray, dark gray and black square represent 0, 1, 2, respectively.

Fig.A1 Plot of solutions of ultradiscrete and difference diffusion equation. (a) F^t_{ij} in (43), (b) $\varepsilon \log f^t_{ij}$ in (46) for $\varepsilon = 0.2$ and (c) $\varepsilon \log f^t_{ij}$ in (46) for $\varepsilon = 0.01$. In all figures, we set $t = 2$ and $i_0 = j_0 = 0$.

Figure 1 by K. Nishinari, et al

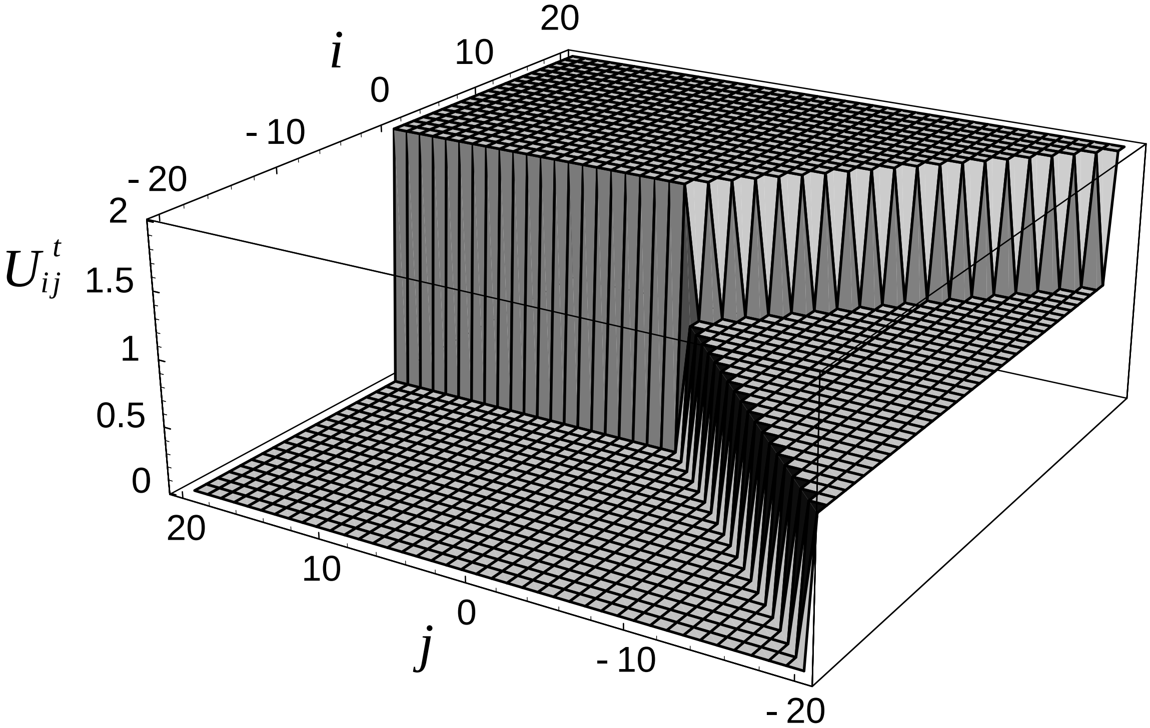

Figure 2 by K. Nishinari, et al

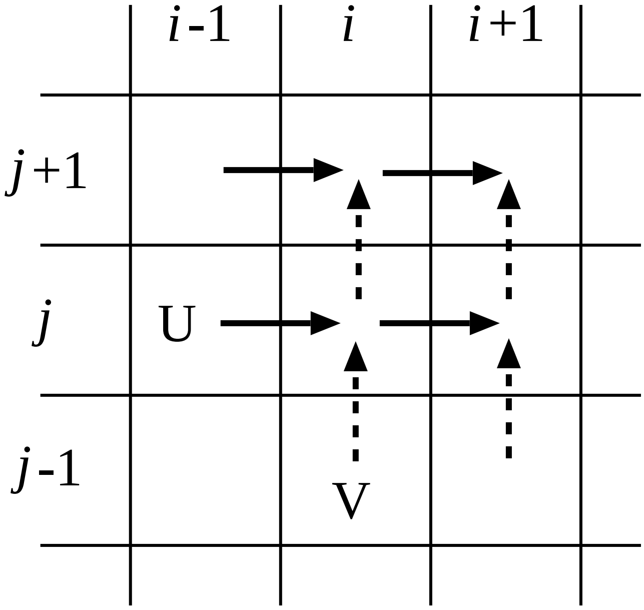

Figure 3 by K. Nishinari, et al

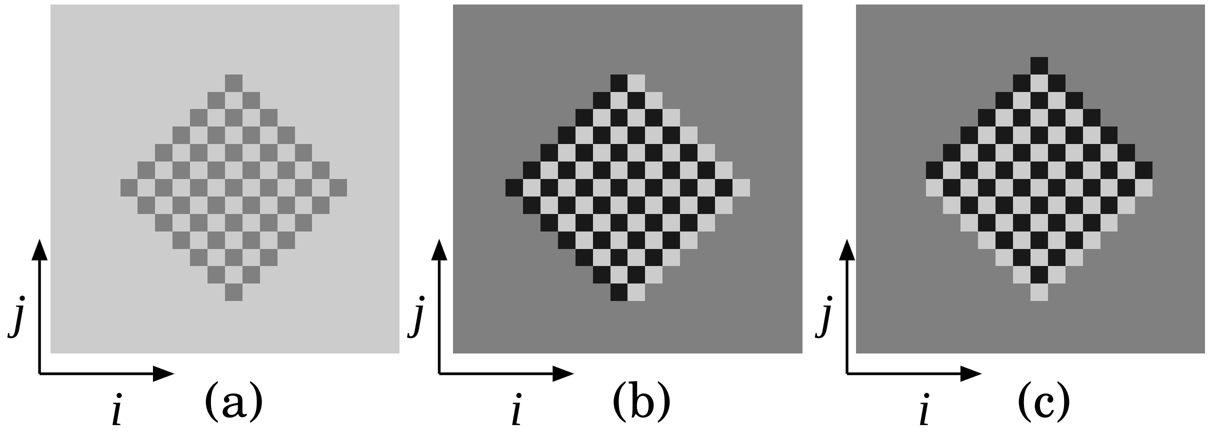

Figure 4 by K. Nishinari, et al

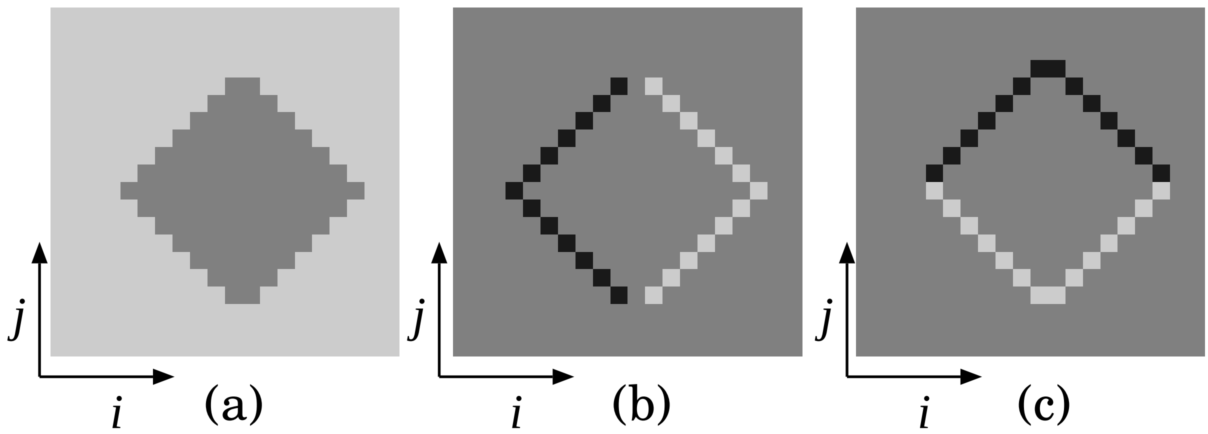

Figure A1 by K. Nishinari, et al