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2+1 dimensional soliton cellular automaton

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Abstract

A 2+1 dimensional soliton cellular automaton is derived from discrete analogue of a generalized Toda equation through the procedure of the so-called ultra-discretization. Its soliton solution and the time evolution are also discussed.

1 Introduction

Recently, discrete soliton systems have attracted much attention. Among them, "ultra-discrete" soliton equations, in which dependent variables as well as independent variables take discrete values, have been actively studied. One of the most important ultra-discrete soliton systems is the so-called "soliton cellular automaton", or SCA for short [1]. This is 1(space) + 1(time) dimensional and two-valued (0 and 1). The time evolution of the value of the $j-$th cell at time t, u_j^t, is given by

$$u_j^{t+1} = \begin{cases} 1 & \text{if } u_j^t = 0 \ \& \ \sum_{i=-\infty}^{j-1} u_i^t > \sum_{i=-\infty}^{j-1} u_i^{t+1} \\ 0 & \text{otherwise} \end{cases} \qquad (1)$$

A remarkable feature of equation (1) is that any state consists only of solitons, interacting in the same manner as KdV solitons. Moreover it possesses an abundant combinatoric structure and an infinite number of conserved quantities [2]. Quite recently, a direct connection between the SCA and the Lotka-Volterra equation, which is considered as one integrable discretization of the KdV equation, has been clarified [3, 4]. A key to the discretization, which we call the "ultra-discretization" in this context, is the following formula:

$$\lim_{\varepsilon\to+0} \varepsilon \log(1 + \mathrm{e}^{X/\varepsilon}) = F(X) = \max[0, X]. \qquad (2)$$

The purpose of this paper is to present a 2+1 dimensional SCA derived from ultra-discretization of the 2+1 dimensional Toda equation. In section 2, we derive the 2+1 dimensional SCA through ultra-discretization of a special case of DAGTE, i.e. discrete analogue of a generalized Toda equation. In section 3, we discuss soliton solution for the 2+1 dimensional SCA and its time evolution. Concluding remarks are given in section 4.

2 Discrete analogue of a generalized Toda equation and the 2+1 dimensional SCA

We start with the following difference equation proposed by Hirota [5],

$$[Z_1 \exp(D_1) + Z_2 \exp(D_2) + Z_3 \exp(D_3)]\, f \cdot f = 0, \qquad (3)$$

where $Z_i (i = 1, 2, 3)$ are arbitrary parameters and $D_i (i = 1, 2, 3)$ stand for Hirota's derivatives [6] with respect to variables of the unknown function f. Equation (3) is called discrete analogue of a generalized Toda equation, or Hirota-Miwa equation. Many soliton equations are obtained by taking proper limit of equation (3) [5].

In this paper, we consider a particular case of equation (3),

$$\left\{\exp(D_t) - \delta^2 \exp(D_x) - (1 - \delta^2) \exp(D_y)\right\} f \cdot f = 0, \qquad (4)$$

or equivalently,

$$\begin{aligned} &f(t-1, x, y) f(t+1, x, y) - \delta^2 f(t, x-1, y) f(t, x+1, y) \\ &\quad -(1-\delta^2) f(t, x, y+1) f(t, x, y-1) = 0. \end{aligned} \qquad (5)$$

Equation (5) reduces to a discrete analogue of the 2+1 dimensional Toda equation [7, 8],

$$\begin{cases} V(l+1,m,n) - V(l,m,n) = I(l,m+1,n)V(l+1,m,n) \\ \qquad\qquad -I(l,m,n+1)V(l,m,n), \\ I(l,m+1,n) - I(l,m+1,n) = V(l+1,m,n-1) - V(l,m,n). \end{cases} \tag{6}$$

through the following independent and dependent variable transformations,

$$l = \frac{-x-y-t+1}{2}, \quad m = \frac{x-y+t+1}{2}, \quad n = x \tag{7}$$

$$V(l,m,n) = \frac{\tau(l+1,m,n+1)\tau(l,m+1,n-1)}{\tau(l,m+1,n)\tau(l,m,n)} \tag{8}$$

$$I(l,m,n) = \frac{1}{\delta}\left\{(1-\delta^2)\frac{\tau(l+1,m,n)\tau(l,m,n-1)}{\tau(l,m,n)\tau(l+1,m,n-1)} - 1\right\} \tag{9}$$

$$\tau(l+\frac{1}{2}, m+\frac{1}{2}, n) = f(x,y,z). \tag{10}$$

Let us derive an ultra-discrete version of equation (5). The dependent variable transformation,

$$f(t,x,y) = \exp[S(t,x,y)] \tag{11}$$

yields

$$\exp[\Delta_t^2 S(t,x,y)] - \delta^2 \exp[\Delta_x^2 S(t,x,y)] - (1-\delta^2)\exp[\Delta_y^2 S(t,x,y)] = 0, \tag{12}$$

or equivalently,

$$\exp[(\Delta_t^2 - \Delta_y^2)S(t,x,y)] = (1-\delta^2)\left(1 + \frac{\delta^2}{1-\delta^2}\exp[(\Delta_x^2 - \Delta_y^2)S(t,x,y)]\right). \tag{13}$$

Each operator Δ_t, Δ_x and Δ_y represents central difference operator defined, for example, by

$$\Delta_t^2 S(t,x,y) = S(t+1,x,y) - 2S(t,x,y) + S(t-1,x,y). \tag{14}$$

Taking a logarithm of equation (13) and operating $(\Delta_x^2 - \Delta_y^2)$, we have

$$(\Delta_t^2 - \Delta_y^2)u(t,x,y) = (\Delta_x^2 - \Delta_y^2)\log\left(1 + \frac{\delta^2}{1-\delta^2}\exp[u(t,x,y)]\right), \tag{15}$$

where

$$u(t,x,y) = (\Delta_x^2 - \Delta_y^2)S(t,x,y). \tag{16}$$

We finally take an ultra-discrete limit of equation (15). Putting

$$u(t,x,y)=\frac{v_\varepsilon(t,x,y)}{\varepsilon},\ \frac{\delta^2}{1-\delta^2}=\mathrm{e}^{-\frac{\theta_0}{\varepsilon}}, \tag{17}$$

and taking the small limit of ε, we obtain the following equation,

$$(\Delta_t^2-\Delta_y^2)v(t,x,y) = (\Delta_x^2-\Delta_y^2)F(v(t,x,y)-\theta_0), \tag{18}$$

$$F(X) = \max[0,X]. \tag{19}$$

We have rewritten $\lim_{\varepsilon\to+0} v_\varepsilon(t,x,y)$ as $v(t,x,y)$ in equation (18). We call the ultra-discrete system satisfying the above equation (18) the *2+1 dimensional SCA*.

3 Soliton solution for the 2+1 dimensional SCA

In this section, we discuss soliton solution for the 2+1 dimensional SCA governed by equation (18). Since we have derived equation (18) by taking an ultra-discrete limit of bilinear equation (5), we may well consider that the soliton solution for equation (18) is also obtained by ultra-discretization of that for equation (5).

We first consider one-soliton solution. The bilinear equation (5) admits one-soliton solution given by

$$f(t,x,y)=1+\mathrm{e}^{\eta},\ \eta=px+qy+\omega t, \tag{20}$$

where the set of parameters (p,q,ω) satisfies a dispersion relation,

$$(\mathrm{e}^{-\omega}+\mathrm{e}^{\omega})-\delta^2(\mathrm{e}^{-p}+\mathrm{e}^{p})-(1-\delta^2)(\mathrm{e}^{-q}+\mathrm{e}^{-q})=0. \tag{21}$$

Following the procedures given by eqs. (11) and (16), we have

$$u(t,x,y)=\log(1+\mathrm{e}^{\eta+p})+\log(1+\mathrm{e}^{\eta-p})-\log(1+\mathrm{e}^{\eta+q})-\log(1+\mathrm{e}^{\eta-q}). \tag{22}$$

In order to take an ultra-discrete limit, we introduce new variables as

$$\varepsilon p=P,\ \varepsilon q=Q,\ \varepsilon\omega=\Omega,\ K=Px+Qy+\Omega t, \tag{23}$$

$$v_\varepsilon(t,x,y)=\varepsilon u(t,x,y). \tag{24}$$

Taking a limit $\varepsilon\to+0$, we obtain

$$v(t,x,y)=F(K+P)+F(K-P)-F(K+Q)-F(K-Q). \tag{25}$$

The dispersion relation (21) reduces, through the same limiting procedure, to

$$|\Omega| = \max[|P|, |Q| + \theta_0] - \max[0, \theta_0]. \tag{26}$$

Next we construct two-soliton solution. Equation (5) possesses two-soliton solution written as

$$f(t,x,y) = 1 + e^{\eta_1} + e^{\eta_2} + e^{\eta_1+\eta_2+\theta_{12}},\ \eta_i = p_i x + q_i y + \omega_i t\ (i = 1,2), \tag{27}$$

$$(e^{-\omega_i} + e^{\omega_i}) - \delta^2(e^{-p_i} + e^{p_i}) - (1-\delta^2)(e^{-q_i} + e^{-q_i}) = 0\ (i = 1,2). \tag{28}$$

The variable θ_{12} stands for a phase shift and is determined by the following relation:

$$e^{\theta_{12}} = -\frac{(e^{-\omega_1+\omega_2} + e^{\omega_1-\omega_2}) - \delta^2(e^{-p_1+p_2} + e^{p_1-p_2}) - (1-\delta^2)(e^{-q_1+q_2} + e^{q_1-q_2})}{(e^{\omega_1+\omega_2} + e^{-\omega_1-\omega_2}) - \delta^2(e^{p_1+p_2} + e^{-p_1-p_2}) - (1-\delta^2)(e^{q_1+q_2} + e^{-q_1-q_2})}. \tag{29}$$

Introducing new variables as

$$\varepsilon p_i = P_i,\ \varepsilon q_i = Q_i,\ \varepsilon\omega_i = \Omega_i,\ K_i = P_i x + Q_i y + \Omega_i t,\ (i = 1,2) \tag{30}$$

$$v_\varepsilon(t,x,y) = \varepsilon u(t,x,y),\ \varepsilon\theta_{12} = \Theta_{12}, \tag{31}$$

and taking the same limit $\varepsilon \to +0$, we have

$$\begin{aligned} v(t,x,y) &= \max[0, K_1+P_1, K_2+P_2, K_1+K_2+P_1+P_2+\Theta_{12}] \\ &\quad + \max[0, K_1-P_1, K_2-P_2, K_1+K_2-P_1-P_2+\Theta_{12}] \\ &\quad - \max[0, K_1+Q_1, K_2+Q_2, K_1+K_2+Q_1+Q_2+\Theta_{12}] \\ &\quad - \max[0, K_1-Q_1, K_2-Q_2, K_1+K_2-Q_1-Q_2+\Theta_{12}] \quad (32) \\ |\Omega_i| &= \max[|P_i|, |Q_i| + \theta_0] - \max[0, \theta_0]\ (i = 1,2). \quad (33) \end{aligned}$$

Phase shift term Θ_{12} is determined by

$$\begin{aligned} &\max[\Theta_{12} + \max[0,\theta_0] + |\Omega_1 + \Omega_2|, \max[0,\theta_0] + |\Omega_1 - \Omega_2|] \\ &= \max[\Theta_{12} + |P_1 + P_2|, \Theta_{12} + \theta_0 + |Q_1 + Q_2|, \\ &\qquad |P_1 - P_2|, \theta_0 + |Q_1 - Q_2|], \end{aligned} \tag{34}$$

which is obtained through the same limit $\varepsilon \to +0$ in equation (29).

It should be noted that N-soliton solution can also be found through the same limiting procedure. This is given by

$$\begin{aligned} v(t,x,y) &= \max_{\mu=0,1}\left[\max_{i=1,2,\cdots,N}[\mu_i(K_i + P_i)], \max_{i<j}[\mu_i\mu_j\Theta_{ij}]\right] \\ &\quad + \max_{\mu=0,1}\left[\max_{i=1,2,\cdots,N}[\mu_i(K_i - P_i)], \max_{i<j}[\mu_i\mu_j\Theta_{ij}]\right] \end{aligned}$$

$$
\begin{aligned}
& - \max_{\mu=0,1}\left[\max_{i=1,2,\cdots,N}[\mu_i(K_i+Q_i)], \max_{i<j}[\mu_i\mu_j\Theta_{ij}]\right] \\
& - \max_{\mu=0,1}\left[\max_{i=1,2,\cdots,N}[\mu_i(K_i-Q_i)], \max_{i<j}[\mu_i\mu_j\Theta_{ij}]\right], \qquad (35) \\
& K_i = P_i x + Q_i y + \Omega_i t, \\
|\Omega_i| \ = \ & \max[|P_i|, |Q_i| + \theta_0] - \max[0, \theta_0]. \qquad (36)
\end{aligned}
$$

Each phase shift term $\Theta_{ij} (1 \leq i < j \leq N)$ satisfies the relation,

$$
\begin{aligned}
& \max\left[\Theta_{ij} + \max[0,\theta_0] + |\Omega_i + \Omega_j|, \max[0,\theta_0] + |\Omega_i - \Omega_j|\right] \\
= \ & \max\left[\Theta_{ij} + |P_i + P_j|, \Theta_{ij} + \theta_0 + |Q_i + Q_j|,\right. \\
& \left. |P_i - P_j|, \theta_0 + |Q_i - Q_j|\right]. \qquad (37)
\end{aligned}
$$

4 Concluding Remarks

We have shown that a 2+1 dimensional SCA is obtained by taking an ultra-discrete limit of the 2+1 dimensional Toda equation. We have also found its soliton solutions. It is a future problem to construct an ultra-discrete version of other kind of solutions, for example, rational, molecular and quasi-periodic solutions.

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A. Time evolutions of one- and two-solitons

We here show one- and two-soliton solutions and their time evolutions in the following figures. Figure 1 displays one soliton solution with parameters $P = 5, Q = 1$ and $\theta_0 = 2$. Figures 2 and 3 demonstrate two-soliton solution at $t = -4, -3$ and its time evolution, respectively, with parameters $P_1 = 6, Q_1 = 1, P_2 = 6, Q_2 = 5$ and $\theta_0 = 2$.

Figures 2 and 3 demonstrate two-soliton solution at $t = -4, -3$ and its time evolution, respectively, with parameters $P_1 = 6$, $Q_1 = 1$, $P_2 = 6$, $Q_2 = 5$ and $\theta_0 = 2$.

```
 8  0000032000000000
 7  0000023000000000
 6  0000014000000000
 5  0000004000000000
 4  0000004100000000
 3  0000003200000000
 2  0000002300000000
 1  0000001400000000
 0  0000000400000000    ==>
-1  0000000410000000
-2  0000000320000000
-3  0000000230000000
-4  0000000140000000
-5  0000000040000000
-6  0000000041000000
-7  0000000032000000
 y  x.......012345678
```

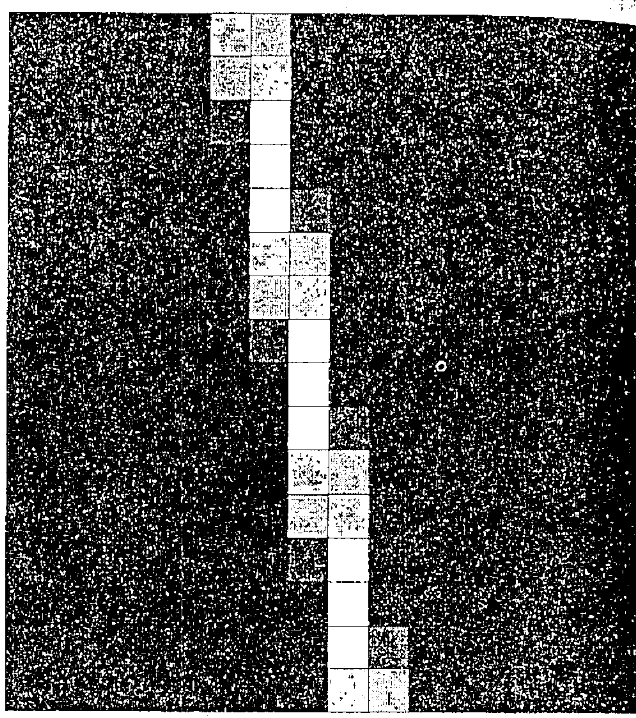

Figure 1: One-solition solution ($t = 0$) (The horizontal and vertical axes represent x and y coordinates, respectively. At the bottom of the left-side figure, negative values of x coordinate are expressed as "." for convenience' sake.)

```
                t=-4
 15 000000000000004200000000000000
 14 000000000000003300000000000000
 13 100000000000002400000000000000
 12 110000000000001500000000000000
 11 011000000000000500000000000000
 10 001100000000000510000000000000
  9 000110000000000420000000000000
  8 000010000000000330000000000000
  7 000011000000000240000000000000
  6 000001100000000150000000000000
  5 000000110000000050000000000000
  4 000000011000000051000000000000
  3 000000001100000042000000000000
  2 000000000100000033000000000000
  1 000000000110000024000000000000
  0 000000000011000015000000000000
 -1 000000000001100005000000000000
 -2 000000000000110005100000000000
 -3 000000000000011004200000000000
 -4 000000000000001003300000000000
 -5 000000000000001102400000000000
 -6 000000000000000111500000000000
 -7 000000000000000011500000000000
 -8 000000000000000001610000000000
 -9 000000000000000000520000000000
-10 000000000000000000431000000000
-11 000000000000000000331000000000
-12 000000000000000000241100000000
-13 000000000000000000150110000000
-14 000000000000000000050011000000
   yx..............0123456789******
```

```
                t=-3
 15 000000000000024000000000000000
 14 100000000000015000000000000000
 13 110000000000005000000000000000
 12 011000000000005100000000000000
 11 001100000000004200000000000000
 10 000110000000003300000000000000
  9 000010000000002400000000000000
  8 000011000000001500000000000000
  7 000001100000000500000000000000
  6 000000110000000510000000000000
  5 000000011000000420000000000000
  4 000000001100000330000000000000
  3 000000000100000240000000000000
  2 000000000110000150000000000000
  1 000000000011000050000000000000
  0 000000000001100051000000000000
 -1 000000000000110042000000000000
 -2 000000000000011033000000000000
 -3 000000000000001024000000000000
 -4 000000000000001115000000000000
 -5 000000000000000115000000000000
 -6 000000000000000016100000000000
 -7 000000000000000005200000000000
 -8 000000000000000004310000000000
 -9 000000000000000003311000000000
-10 000000000000000002401000000000
-11 000000000000000001501100000000
-12 000000000000000000500110000000
-13 000000000000000000510011000000
-14 000000000000000000420001100000
   yx..............0123456789******
```

Figure 2: Two-solition solution ($t = -4, -3$) (At the bottom, negative values of x coordinate are expressed as "."and values greater than 10 are also done as "*" for convenience sake.)

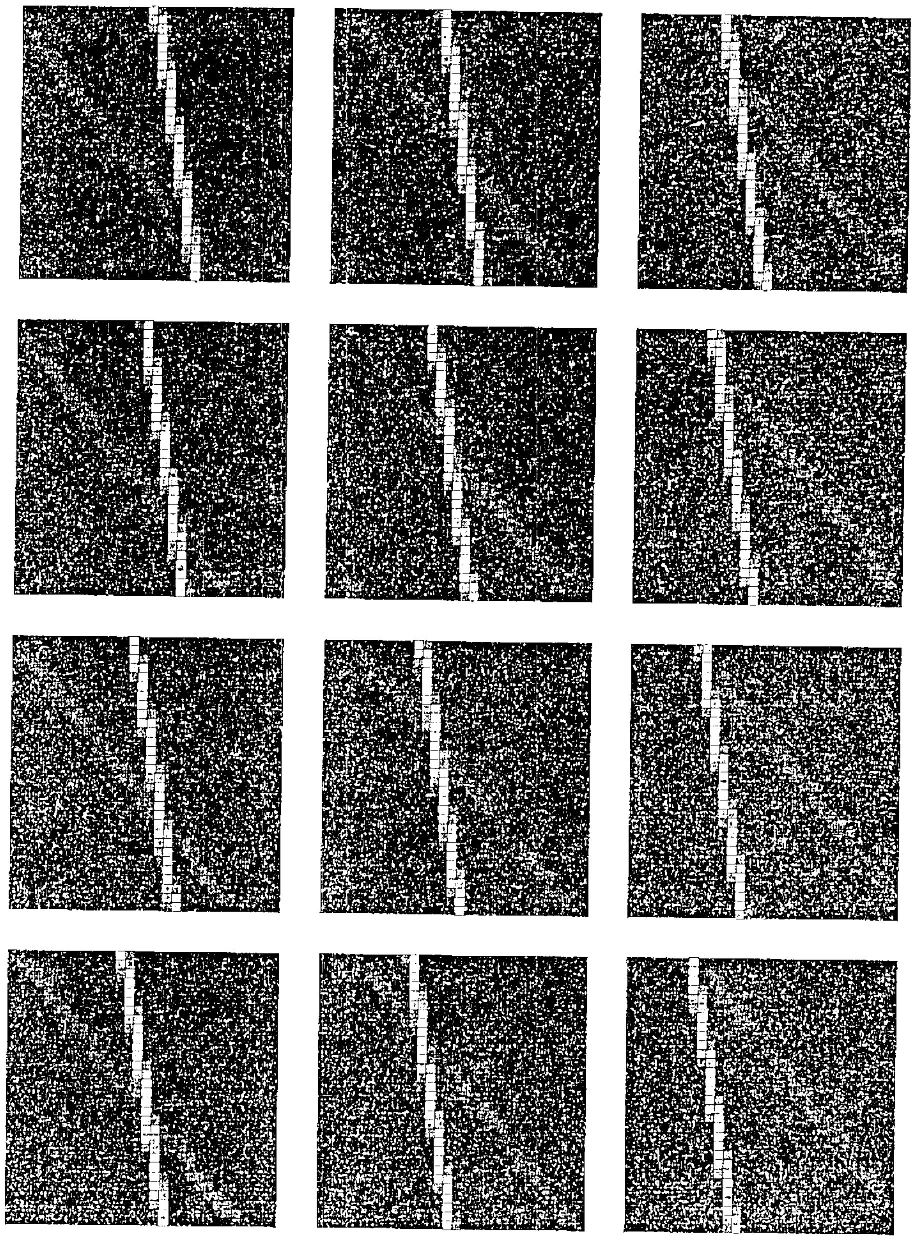

Figure 3: Time evolution of two-soliton solution (Four left-side figures display, from top to bottom, values of $v(t, x, y)$ at $t = -4, -3, -2. -1$. four central figures at $t = 0, 1, 2, 3$ and four right-hand figures at $t = 4, 5, 6, 7$.)