Toda-type Cellular Automaton and its N-soliton Solution

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Abstract

In this letter, we show that the cellular automaton proposed by two of the authors (D.T and J.M) is obtained from the discrete Toda lattice equation through a special limiting procedure. Also by applying a similar kind of limiting procedure to the N-soliton solution of the discrete Toda lattice equation, we obtain the N-soliton solution for this cellular automaton.

Keywords: Soliton; Discrete; Cellular Automaton; Nonlinear; Toda Lattice

The phenomena we observe in nature have been described in many ways. Among several methods to analyze the behavior of nature, differential equations have been traditionally the most powerful and often used. However many systems in the fields of biology, statistical physics, etc., are difficult to describe using differential equations. These systems are rather easy to deal with using discrete methods, such as discrete equations, coupled map lattices and cellular automata(CA's)[0].

Due to the enormous growth of computer power in recent years, we have been able to analyze these discrete systems even though they have large degrees of freedom and strong nonlinearity. Among them, CA's are most suited for computer simulation because all of the variables are discrete, including field variables, and round off error does not occur. Therefore CA's are extensively studied and various statistical results are obtained, though traditional methods used in differential calculus could not have been applied due to the strong nonlinearity.

On the other hand, in the field of nonlinear physics, soliton theory has succeeded as an analytical tool for nonlinear evolution equations for almost 30

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years and has been applied to several fields: hydrodynamics, plasma physics, optical physics and so on. Moreover recent development of soliton theory tells us it can be also applicable to discrete equations[0].

The notion of soliton for CA's was first introduced by Park et al[0]. After this work, soliton-like structures have been found in several CA's and attempts to apply soliton theory to CA's have been made by several groups [0,0,0,0]. However direct relation of CA to soliton equations has not been clear.

Recently we proposed a general method to obtain CA's from discrete soliton equations through a limiting procedure[0]. By using this method, we clarified the relation between the CA which was proposed by two of the authors (D.T. and J.S.)[0] and the Korteweg de-Vries equation. This is our answer to one of the unsolved problems listed in the paper of Wolfram[0].

In this letter, we apply this method to the Toda lattice equation. We show that the CA, which is proposed in the previous paper[0], is obtained from the discrete Toda lattice equation through the limiting procedure. Also we obtain N-soliton solutions of this CA from those of the discrete Toda lattice equation.

The starting point is the discrete Toda lattice equation which was introduced by Hirota[0],

$$\begin{aligned}&\log(1+V_n^{t+1}) - 2\log(1+V_n^t) + \log(1+V_n^{t-1})\\&= \log(1+\delta^2 V_{n+1}^t) - 2\log(1+\delta^2 V_n^t) + \log(1+\delta^2 V_{n-1}^t).\end{aligned} \tag{1}$$

By introducing $V_n^t = e^{U_n^t} - 1$, we obtain

$$\begin{aligned}&U_n^{t+1} - 2U_n^t + U_n^{t-1}\\&= \log(1+\delta^2(e^{U_{n+1}^t}-1)) - 2\log(1+\delta^2(e^{U_n^t}-1)) + \log(1+\delta^2(e^{U_{n-1}^t}-1)).\end{aligned} \tag{2}$$

One can easily obtain the continuous Toda lattice equation,

$$\frac{d^2 r_n}{dt^2} = e^{r_{n+1}} - 2e^{r_n} + e^{r_{n-1}}, \tag{3}$$

from Eq.(2) by the relation $U_n^t = r_n(\delta t)$ and taking $\delta \to 0$.

The N-soliton solution of Eq.(2) is given by

$$U_n^t = \Delta_n^2 \log f_n^t, \tag{4}$$

with

$$f_n^t = \sum_{\mu_i=0,1} \exp[\sum_{i=1}^{N} \mu_i \xi_i + \sum_{i<j}^{(N)} \mu_i \mu_j A_{ij}], \tag{5}$$

where the difference operator Δ_n^2 on F_n is defined by

$$\Delta_n^2 F_n = F_{n+1} - 2F_n + F_{n-1}, \tag{6}$$

and

$$\xi_i = P_i n - \Omega_i t + \xi_i^0, \tag{7}$$
$$\delta^{-1} \sinh(\Omega_i/2) = \sigma_i \sinh(P_i/2), \tag{8}$$
$$\exp A_{ij} = \frac{\sigma_i \sigma_j - \cosh(\frac{P_i+\Omega_i-P_j-\Omega_j}{2})}{\sigma_i \sigma_j - \cosh(\frac{P_i+\Omega_i+P_j+\Omega_j}{2})}, \tag{9}$$
$$\sigma_i, \sigma_j = 1 \quad \text{or} \quad -1. \tag{10}$$

Here ξ_i^0 and $P_i (i = 1, 2, \cdots, N)$ are arbitrary parameters, and $\sum_{\mu_i=0,1}$ denotes the summation over all terms obtained by replacing each μ_i by 0 or 1 and $\sum_{i>j}^{(N)}$ denotes the summation over all possible pairs chosen from N elements.

Now we introduce a positive parameter ϵ defined by $\delta = e^{-\frac{L}{2\epsilon}}$ where L is a positive integer, and set $U_n^t = u_n^t/\epsilon$. Then noticing the fact

$$\lim_{\epsilon \to +0} \epsilon \log(1 + e^{\frac{X}{\epsilon}}) = \max(0, X) = \begin{cases} X & \text{if } X \geq 0, \\ 0 & \text{else,} \end{cases} \tag{11}$$

we obtain from Eq.(2) in the limit $\epsilon \to +0$

$$\begin{aligned} &u_n^{t+1} - 2u_n^t + u_n^{t-1} \\ &= \max(0, u_{n+1}^t - L) - 2\max(0, u_n^t - L) + \max(0, u_{n-1}^t - L), \end{aligned} \tag{12}$$

where the equivalent equation was proposed in the previous paper[0]. Eq.(12) thus obtained is considered to be Toda-type CA, which shares the common algebraic properties with Toda lattice equation as shown below.

Let us look at whether the N-soliton solution can survive by this limiting procedure. The one-soliton solution of Eq.(2) is expressed by

$$U_n^t = \Delta_n^2 \log(1 + e^{Pn - \Omega t + \xi^0}), \tag{13}$$

where

$$\sinh(\Omega/2) = \sigma\delta\sinh(P/2), \tag{14}$$
$$\sigma = 1 \text{ or } -1, \tag{15}$$

Here we set $P = p/\epsilon$, $\Omega = \omega/\epsilon$, and $\xi^0 = \eta^0/\epsilon$, then we obtain

$$u_n^t = \epsilon\Delta_n^2 \log(1 + e^{\frac{pn-\omega t+\eta^0}{\epsilon}}). \tag{16}$$

Taking $\epsilon \to +0$, Eqs.(16) and (14), respectively, become

$$u_n^t = \Delta_n^2 \max(0, pn - \omega t + \eta^0), \tag{17}$$
$$= \begin{cases} 0 & \text{if } n \le \frac{\omega t-\eta^0}{p} - 1, \\ |p|(n+1) - \omega t + \eta^0 & \text{if } \frac{\omega t-\eta^0}{p} - 1 \le n \le \frac{\omega t-\eta^0}{p}, \\ -|p|(n-1) - \omega t + \eta^0 & \text{if } \frac{\omega t-\eta^0}{p} \le n \le \frac{\omega t-\eta^0}{p} + 1, \\ 0 & \text{if } n \ge \frac{\omega t-\eta^0}{p} + 1, \end{cases} \tag{18}$$

and

$$\omega = \begin{cases} \sigma(p-L) & \text{if} \quad p > L, \\ 0 & \text{if } -L \le p \le L, \\ -\sigma(-p-L) & \text{if} \quad p < -L, \end{cases} \tag{19}$$
$$= \sigma(\max(0, p-L) - \max(0, -p-L)). \tag{20}$$

This is the one-soliton solution of Eq.(12), which is identical to the one shown in the previous paper if we take p, η^0 as integers and $L = 1$. It is easy to see the speed and the maximum amplitude of soliton is expressed by ω/p and $|p|$ from Eq.(18) respectively.

By setting $P_i = p_i/\epsilon$, $\Omega_i = \omega_i/\epsilon, \xi_i^0 = \eta_i^0/\epsilon$, $A_{ij} = a_{ij}/\epsilon$ and noticing the fact

$$\lim_{\epsilon\to+0} \epsilon \log(\sum_{i=1}^{M} e^{\frac{X_i}{\epsilon}}) = \max(X_1, X_2, \ldots, X_{M-1}, X_M), \tag{21}$$

we also obtain the N-soliton solution in the limit $\epsilon \to +0$

$$u_n^t = \Delta_n^2 \rho_n^t, \tag{22}$$

with

$$\rho_n^t = \max_{\mu_i=0,1}[\sum_{i=1}^{N}\mu_i\eta_i + \sum_{i<j}^{(N)}\mu_i\mu_j a_{ij}], \tag{23}$$

where $\eta_i = p_i n - \omega_i t + \eta_i^0$, and

$$\omega_i = \sigma_i(\max(0, p_i - L) - \max(0, -p_i - L)), \tag{24}$$

$$a_{ij} = \begin{cases} -2\min(|p_i|, |p_j|) + L, & \text{if} \quad \sigma_i = -1 \text{ and } \sigma_j = -1, \\ \max(\min(p_i + \omega_i, -p_j - \omega_j), \min(-p_i - \omega_i, p_j + \omega_j)), & \text{else.} \end{cases} \tag{25}$$

(About precise derivation of Eq.(25), see Appendix.) Here p_i, η_i^0 are arbitrary parameters, and $\max_{\mu_i=0,1}[X(\mu_i)]$ denotes the maximum value in 2^N possible values of $X(\mu_i)$ obtained by replacing each μ_i by 0 or 1. This solution expresses the interaction of solitons as shown in the previous paper if we take p_i, η_i^0 as integers and $L = 1$.

Let us see how the phase shift of solitons are calculated from above formula by taking 2-soliton solution's case as example. Consider 2-soliton solution

$$\rho_n^t = \max(0, \eta_1, \eta_2, \eta_1 + \eta_2 + a_{12}), \tag{26}$$

with

$$\begin{aligned} &p_1 > p_2 \geq L \geq 1, \\ &\omega_1 = p_1 - L,\ \omega_2 = p_2 - L, \\ &\eta_1^0 = 0,\ \eta_2^0 = 0, \end{aligned} \tag{27}$$

¿From Eq.(27) and Eq.(25), Eq.(26) is written by

$$\begin{aligned} \rho_n^t = &\max(0, p_1 n - (p_1 - L)t, p_2 n - (p_2 - L)t, \\ &(p_1 + p_2)n - (p_1 + p_2 - 2L)t - (2p_2 - L)). \end{aligned} \tag{28}$$

Fig.1 show the interaction of solitons where we take $p_1 = 3$, $p_2 = 2$ and $L = 1$. At the time $t = -\infty$ and around the region $\eta_1 \approx 0$, i.e. $n \approx \frac{p_1 - L}{p_1}t$, we have

$$\eta_2 \approx \frac{L(p_1 - p_2)}{p_1}t \to -\infty, \tag{29}$$

and therefore the solution is written as

$$\rho_n^t = \max(0, \eta_1). \tag{30}$$

This expresses that one of the solitons exists around the region $\eta_1 \approx 0$ at $t = -\infty$. Similarly, at the time $t = \infty$ and around the region $\eta_1 \approx 0$, we have

$$\begin{aligned}
\rho_n^t &= \max(0, \eta_1, \eta_2, \eta_1 + \eta_2 + a_{12}), \\
&= \eta_2 + \max(0, \eta_1 + a_{12}, -\eta_2, \eta_1 - \eta_2), &(31)\\
&\asymp \max(0, \eta_1 + a_{12}, -\eta_2, \eta_1 - \eta_2), &(32)\\
&= \max(0, \eta_1 + a_{12}), &(33)
\end{aligned}$$

where $\asymp$ denotes l.h.s and r.h.s give same solution, because the first term of l.h.s η_2 vanishes under the operation of difference operator Δ_n^2. This expresses that the soliton also exists around the region $\eta_1 \approx 0$ at $t = \infty$ but its position is shifted due to the term a_{12}. Similarly around the region $\eta_2 \approx 0$, the solution is expressed by

$$\rho_n^t = \begin{cases} \max(0, \eta_2 + a_{12}) & \text{at} \quad t = -\infty, \\ \max(0, \eta_2) & \text{at} \quad t = \infty, \end{cases} \tag{34}$$

and this expresses the other soliton. The value of the phase shift of this solution in the case of $L = 1$ is given as follows. Eq.(33) can be written

$$\begin{aligned}
\rho_n^t &= \max(0, p_1 n - (p_1 - 1)t - (2p_2 - 1)), &(35)\\
&= \max(0, p_1(n - (2p_2 - 1)) - (p_1 - 1)(t - (2p_2 - 1))), &(36)
\end{aligned}$$

and Eq.(34) at $t = -\infty$ can be written

$$\begin{aligned}
\rho_n^t &= \max(0, p_2 n - (p_2 - 1)t - (2p_2 - 1)), &(37)\\
&= \max(0, p_2(n - 1) - (p_2 - 1)(t + 1)). &(38)
\end{aligned}$$

Therefore the solitons shifts its position

$$\begin{array}{ll} (2p_2 - 1, 2p_2 - 1) & \text{for the soliton near } \eta_1 \approx 0, \\ (-1, 1) & \text{for the soliton near } \eta_2 \approx 0, \end{array} \tag{39}$$

in n-t plain.

Similarly in the case 2-soliton solution with

$$\begin{aligned} &L = 1,\ p_1 > p_2 \geq L, \\ &\omega_1 = -p_1 + 1,\ \omega_2 = -p_2 + 1, \\ &\eta_1^0 = 0,\ \eta_2^0 = 0, \end{aligned} \tag{40}$$

the phase shift is given by

$$\begin{aligned} &(-2p_2 + 1, 2p_2 - 1) \quad && \text{for the soliton near } \eta_1 \approx 0, \\ &(1, 1) && \text{for the soliton near } \eta_2 \approx 0, \end{aligned} \tag{41}$$

(See Fig.2 where we take $p_1 = 3, p_2 = 2$) and in the case with

$$\begin{aligned} &L = 1,\ p_1 \geq L,\ p_2 \geq L, \\ &\omega_1 = p_1 - 1,\ \omega_2 = -p_2 + 1, \\ &\eta_1^0 = 0,\ \eta_2^0 = 0, \end{aligned} \tag{42}$$

the phase shift is given by

$$\begin{aligned} &(1, 1) \quad && \text{for the soliton near } \eta_1 \approx 0, \\ &(-1, 1) && \text{for the soliton near } \eta_2 \approx 0. \end{aligned} \tag{43}$$

(See Fig.3 where we take $p_1 = 3, p_2 = 2$) These coincident with the result of previous paper[0]. However in cases other than $L = 1$, similar estimation cannot always be applied because it can happens that the pattern of soliton changes after the interaction. See Fig.4 where we take $L = 3, p_1 = 9, p_2 = 4, \omega_1 = 6, \omega_2 = 1$. Although we showed only two soliton cases, we can also deal with the interaction of N-soliton. For example, the 4-soliton solution shown by Fig.5 in the previous paper is obtained by setting,

$$\begin{aligned} &p_1 = 7, \quad p_2 = 3, \quad p_3 = 1,\ p_4 = -2,\ L = 1, \\ &\omega_1 = 6, \quad \omega_2 = 2, \quad \omega_3 = 0, \omega_4 = 1, \\ &\eta_1^0 = -10, \eta_2^0 = -1, \eta_3^0 = 0,\ \eta_4^0 = -2. \end{aligned} \tag{44}$$

It can also be shown that the phase shift of the N-soliton solution is given by a summation of that of 2-soliton's, which each soliton have had through the interaction with other solitons.

Finally, it should be noted that ρ_n^t satisfies

$$\rho_n^{t+1} + \rho_n^{t-1} = \max(2\rho_n^t, \rho_{n+1}^t + \rho_{n-1}^t - L), \tag{45}$$

which is obtained from Eqs.(12) and (22), and this equation may be considered an analogue of bilinear identity.

In this paper, we have derived a Toda-type CA from the discrete Toda lattice equation and given a formula for the N-soliton solution. This CA inherits the properties of the Toda lattice equation including solitary waves and soliton interactions. We are currently investigating physical properties, such as conserved quantities[0], and physical meaning, in terms of a dynamical system, and will report our results in forthcoming papers. Also, the algebraic structure of this class of CA's is to be studied in detail, and remains an open question for the future.

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Appendix

In the cases $\sigma_i \neq -1$ or $\sigma_j \neq -1$, arguments of cosh in Eq.(10) do not go to 0 if we take $p_i \neq p_j$. Therefore when taking $\epsilon \to +0$, cosh terms dominate and a_{ij} tend to

$$a_{ij} = \epsilon \log \frac{\sigma_i\sigma_j - \cosh((p_i + \omega_i - p_j - \omega_j)/2\epsilon)}{\sigma_i\sigma_j - \cosh((p_i + \omega_i + p_j + \omega_j)/2\epsilon)}, \tag{A.1}$$

$$\sim \epsilon \log \frac{\exp((p_i + \omega_i - p_j - \omega_j)/2\epsilon) + \exp((-p_i - \omega_i + p_j + \omega_j)/2\epsilon)}{\exp((p_i + \omega_i + p_j + \omega_j)/2\epsilon) + \exp((-p_i - \omega_i - p_j - \omega_j)/2\epsilon)}, \tag{A.2}$$

$$\sim \max\left(\frac{p_i + \omega_i - p_j - \omega_j}{2}, \frac{-p_i - \omega_i + p_j + \omega_j}{2}\right)$$
$$- \max\left(\frac{p_i + \omega_i + p_j + \omega_j}{2}, \frac{-p_i - \omega_i - p_j - \omega_j}{2}\right), \tag{A.3}$$

$$= \max\left(\frac{p_i + \omega_i - p_j - \omega_j}{2} - \max\left(\frac{p_i + \omega_i + p_j + \omega_j}{2}, \frac{-p_i - \omega_i - p_j - \omega_j}{2}\right)\right.,$$

$$\frac{-p_i-\omega_i+p_j+\omega_j}{2} - \max\left(\frac{p_i+\omega_i+p_j+\omega_j}{2}, \frac{-p_i-\omega_i-p_j-\omega_j}{2}\right)\Bigg), \tag{A.4}$$

$$= \max(-\max(p_j+\omega_j, -p_i-\omega_i), -\max(p_i+\omega_i, -p_j-\omega_j)), \tag{A.5}$$

$$= \max(\min(p_i+\omega_i, -p_j-\omega_j), \min(-p_i-\omega_i, p_j+\omega_j)). \tag{A.6}$$

In the case of $\sigma_i = -1$ and $\sigma_j = -1$, we need careful estimation because the numerator or the denominator of Eq.(10) goes to 0.

Let us take $p_i > L$ and $p_j > L$ as an example. From Eq.(8), we have

$$\frac{\omega_i}{2\epsilon} = -arcsinh\left(\exp(-\frac{L}{2\epsilon})\sinh\frac{p_i}{2\epsilon}\right). \tag{A.7}$$

Considering the argument of arcsinh diverges and using the asymptotic expansion $arcsinh x \sim \log 2x + \frac{1}{4x^2}$ for $x \gg 0$, we have

$$\frac{\omega_i}{2\epsilon} \sim -\log\left(2\exp(-\frac{L}{2\epsilon})\sinh\frac{p_i}{2\epsilon}\right) - \frac{1}{4\exp(-L/\epsilon)\sinh^2(p_i/2\epsilon)}. \tag{A.8}$$

Therefore

$$\begin{aligned}\frac{p_i+\omega_i-p_j-\omega_j}{2\epsilon} \sim \frac{p_i}{2\epsilon} - \log(\sinh\frac{p_i}{2\epsilon}) - \frac{1}{4\exp(-L/\epsilon)\sinh^2(p_i/2\epsilon)} \\ -\frac{p_j}{2\epsilon} + \log(\sinh\frac{p_j}{2\epsilon}) + \frac{1}{4\exp(-L/\epsilon)\sinh^2(p_j/2\epsilon)}.\end{aligned} \tag{A.9}$$

Here consider expanding each term in series of exponential function and estimating an asymptotic behavior. First and second term of r.h.s. of Eq.(A.9) become

$$\begin{aligned}\frac{p_i}{2\epsilon} - \log(\sinh\frac{p_i}{2\epsilon}) &= -\log\frac{\exp(p_i/2\epsilon)-\exp(-p_i/2\epsilon)}{2\exp(p_i/2\epsilon)}, \\ = -\log\frac{1-\exp(-p_i/\epsilon)}{2} &\sim \log 2 + \exp(-\frac{p_i}{\epsilon}) + \exp(-2\frac{p_i}{\epsilon}) + \cdots,\end{aligned} \tag{A.10}$$

and third term becomes

$$\begin{aligned}\frac{1}{4\exp(-L/\epsilon)\sinh^2(p_i/2\epsilon)} &\sim \frac{1}{\exp(-L/\epsilon)(\exp(p_i/2\epsilon)-\exp(-p_i/2\epsilon))^2}, \\ \sim \exp(-\frac{p_i-L}{\epsilon}). &\end{aligned} \tag{A.11}$$

Thus we obtain

$$\begin{aligned}\frac{p_i+\omega_i-p_j-\omega_j}{2\epsilon} &\sim \exp(-\frac{p_i}{\epsilon})+\exp(-2\frac{p_i}{\epsilon})+\cdots-\exp(-\frac{p_i-L}{\epsilon})\\ &\quad -\exp(-\frac{p_j}{\epsilon})-\exp(-2\frac{p_j}{\epsilon})-\cdots+\exp(-\frac{p_j-L}{\epsilon}),\\ &\sim \exp(-\frac{p_j-L}{\epsilon})-\exp(-\frac{p_i-L}{\epsilon}). \end{aligned} \quad \text{(A.12)}$$

Substituting this info the cosh of numerator and using Taylor expansion $1-\cosh^2 x = -2\sinh^2 \frac{x}{2} \sim 2\left(\frac{x}{2}\right)^2$ for $x \ll 1$,

$$\begin{aligned} a_{ij} &\sim \epsilon\log\left\{2\left(\frac{p_i+\omega_i-p_j-\omega_j}{4\epsilon}\right)^2\right\}-\epsilon\log\left\{\cosh\frac{p_i+\omega_i+p_j+\omega_j}{2\epsilon}-1\right\},\\ &\sim \epsilon\log\left(\exp(-\frac{p_j-L}{\epsilon})-\exp(-\frac{p_i-L}{\epsilon})\right)^2-\epsilon\log\left\{\cosh\frac{L}{\epsilon}-1\right\},\\ &\sim 2\max(-p_j+L,-p_i+L)-L,\\ &= -2\min(p_i,p_j)+L. \end{aligned} \quad \text{(A.13)}$$

Similarly considering all other possible case, we obtain

$$a_{ij} = -2\min(|p_i|,|p_j|)+L. \quad \text{(A.14)}$$

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·	·	·	·	·	·	·	·	·	·	·	·	·	·	2	·	·	3	1	·	·	·
·	·	·	·	·	·	·	·	·	·	·	·	·	·	1	1	·	·	4	·	·	·
·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	2	·	·	1	3	·	·

Fig. 1. A 2-soliton solution where $p_1 = 3,\ p_2 = 2,\ \omega_1 = 2,\ \omega_2 = 1$. '·' expresses 0.

·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	1	1	·	2	1	·	·
·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	2	·	1	2	·	·	·
·	·	·	·	·	·	·	·	·	·	·	·	·	·	1	1	·	3	·	·	·	·
·	·	·	·	·	·	·	·	·	·	·	·	·	·	2	·	2	1	·	·	·	·
·	·	·	·	·	·	·	·	·	·	·	·	·	1	1	1	2	·	·	·	·	·
·	·	·	·	·	·	·	·	·	·	·	·	·	2	·	3	·	·	·	·	·	·
·	·	·	·	·	·	·	·	·	·	·	·	1	1	2	1	·	·	·	·	·	·
·	·	·	·	·	·	·	·	·	·	·	·	2	1	2	·	·	·	·	·	·	·
·	·	·	·	·	·	·	·	·	·	·	1	1	3	·	·	·	·	·	·	·	·
·	·	·	·	·	·	·	·	·	·	·	2	2	1	·	·	·	·	·	·	·	·
·	·	·	·	·	·	·	·	·	·	1	2	2	·	·	·	·	·	·	·	·	·
·	·	·	·	·	·	·	·	·	·	3	1	1	·	·	·	·	·	·	·	·	·
·	·	·	·	·	·	·	·	·	2	1	2	·	·	·	·	·	·	·	·	·	·
·	·	·	·	·	·	·	·	1	2	1	1	·	·	·	·	·	·	·	·	·	·
·	·	·	·	·	·	·	·	3	·	2	·	·	·	·	·	·	·	·	·	·	·
·	·	·	·	·	·	·	2	1	1	1	·	·	·	·	·	·	·	·	·	·	·
·	·	·	·	·	·	1	2	·	2	·	·	·	·	·	·	·	·	·	·	·	·
·	·	·	·	·	·	3	·	1	1	·	·	·	·	·	·	·	·	·	·	·	·
·	·	·	·	·	2	1	·	2	·	·	·	·	·	·	·	·	·	·	·	·	·
·	·	·	·	1	2	·	1	1	·	·	·	·	·	·	·	·	·	·	·	·	·
·	·	·	·	3	·	·	2	·	·	·	·	·	·	·	·	·	·	·	·	·	·
·	·	·	2	1	·	1	1	·	·	·	·	·	·	·	·	·	·	·	·	·	·

Fig. 2. A 2-soliton solution where $p_1 = 3,\ p_2 = 2,\ \omega_1 = -2,\ \omega_2 = -1.$

·	·	·	2	1	·	·	·	·	·	·	·	·	·	·	1	1	·	·	·	·
·	·	·	·	3	·	·	·	·	·	·	·	·	·	·	2	·	·	·	·	·
·	·	·	·	1	2	·	·	·	·	·	·	·	·	1	1	·	·	·	·	·
·	·	·	·	·	2	1	·	·	·	·	·	·	·	2	·	·	·	·	·	·
·	·	·	·	·	·	3	·	·	·	·	·	·	1	1	·	·	·	·	·	·
·	·	·	·	·	·	1	2	·	·	·	·	·	2	·	·	·	·	·	·	·
·	·	·	·	·	·	·	2	1	·	·	·	1	1	·	·	·	·	·	·	·
·	·	·	·	·	·	·	·	3	·	·	·	2	·	·	·	·	·	·	·	·
·	·	·	·	·	·	·	·	1	2	·	1	1	·	·	·	·	·	·	·	·
·	·	·	·	·	·	·	·	·	2	1	2	·	·	·	·	·	·	·	·	·
·	·	·	·	·	·	·	·	·	·	4	1	·	·	·	·	·	·	·	·	·
·	·	·	·	·	·	·	·	·	1	1	3	·	·	·	·	·	·	·	·	·
·	·	·	·	·	·	·	·	·	2	·	1	2	·	·	·	·	·	·	·	·
·	·	·	·	·	·	·	·	1	1	·	·	2	1	·	·	·	·	·	·	·
·	·	·	·	·	·	·	·	2	·	·	·	·	3	·	·	·	·	·	·	·
·	·	·	·	·	·	·	1	1	·	·	·	·	1	2	·	·	·	·	·	·
·	·	·	·	·	·	·	2	·	·	·	·	·	·	2	1	·	·	·	·	·
·	·	·	·	·	·	1	1	·	·	·	·	·	·	·	3	·	·	·	·	·
·	·	·	·	·	·	2	·	·	·	·	·	·	·	·	1	2	·	·	·	·
·	·	·	·	·	1	1	·	·	·	·	·	·	·	·	·	2	1	·	·	·
·	·	·	·	·	2	·	·	·	·	·	·	·	·	·	·	·	3	·	·	·
·	·	·	·	1	1	·	·	·	·	·	·	·	·	·	·	·	1	2	·	·

Fig. 3. A 2-soliton solution where $p_1 = 3,\ p_2 = 2,\ \omega_1 = 2,\ \omega_2 = -1$

·	·	·	9	·	·	·	·	4	·	·	·	·	·	·	·	·	·	·	·	·
·	·	·	3	6	·	·	·	3	1	·	·	·	·	·	·	·	·	·	·	·
·	·	·	·	6	3	·	·	2	2	·	·	·	·	·	·	·	·	·	·	·
·	·	·	·	·	9	·	·	1	3	·	·	·	·	·	·	·	·	·	·	·
·	·	·	·	·	3	6	·	·	4	·	·	·	·	·	·	·	·	·	·	·
·	·	·	·	·	·	6	3	·	3	1	·	·	·	·	·	·	·	·	·	·
·	·	·	·	·	·	·	9	·	2	2	·	·	·	·	·	·	·	·	·	·
·	·	·	·	·	·	·	3	6	1	3	·	·	·	·	·	·	·	·	·	·
·	·	·	·	·	·	·	·	6	3	4	·	·	·	·	·	·	·	·	·	·
·	·	·	·	·	·	·	·	·	9	3	1	·	·	·	·	·	·	·	·	·
·	·	·	·	·	·	·	·	·	3	8	2	·	·	·	·	·	·	·	·	·
·	·	·	·	·	·	·	·	·	2	3	8	·	·	·	·	·	·	·	·	·
·	·	·	·	·	·	·	·	·	1	3	4	5	·	·	·	·	·	·	·	·
·	·	·	·	·	·	·	·	·	·	4	·	7	2	·	·	·	·	·	·	·
·	·	·	·	·	·	·	·	·	·	3	1	1	8	·	·	·	·	·	·	·
·	·	·	·	·	·	·	·	·	·	2	2	·	4	5	·	·	·	·	·	·
·	·	·	·	·	·	·	·	·	·	1	3	·	·	7	2	·	·	·	·	·
·	·	·	·	·	·	·	·	·	·	·	4	·	·	1	8	·	·	·	·	·
·	·	·	·	·	·	·	·	·	·	·	3	1	·	·	4	5	·	·	·	·
·	·	·	·	·	·	·	·	·	·	·	2	2	·	·	·	7	2	·	·	·
·	·	·	·	·	·	·	·	·	·	·	1	3	·	·	·	1	8	·	·	·
·	·	·	·	·	·	·	·	·	·	·	·	4	·	·	·	·	4	5	·	·

Fig. 4. A 2-soliton solution for the case $L = 3$