FROM INTEGRABILITY TO CHAOS IN A LOTKA-VOLTERRA CELLULAR-AUTOMATON

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Abstract

We present a cellular automaton equivalent for the 2-dimensional Lotka-Volterra system. The dynamics are studied for integer and rational values of the parameters. In the case of integer parameters the motion is perfectly regular leading to strictly periodic motion. This is *still* true in the case of rational parameters, but for rational initial conditions the period becomes progressively longer as the denominator of the initial data increases. The motion, in this case, loses progressively its regularity resulting to chaotic behavior at the limit of irrational data.

1. Introduction

Integrability and chaos are the two most important manifestations of nonlinear dynamics. Since they are antithetical notions, their study is a highly specialized one, to the point that little attention is paid to the transition from one domain to the other. Moreover, while the study of integrability is performed mainly with analytical tools, chaos is often studied through numerics. This last approach is more often than not questionable: the numerical approximations and the inaccuracies of the numerical treatment may alter the qualitative behaviour of the system, unless special precautions are taken.

Given these difficulties, how can one study the integrability-to-chaos transition in a reliable way? The first step, when one wishes to study numerically a set of continuous equations of motion, is to discretise them. This discretisation in principle destroys integrability and may thus introduce chaos that is not present in the initial problem. The remedy to this, whenever one starts with an integrable continuous problem, is to implement an integrable discretisation [1,2]. The recently developed technique of singularity confinement [3] allows one to check the integrability of the discretisation scheme. Even when one has an integrable numerical scheme, this is not an absolute guarantee because of the round-off errors in the implementation of the algorithm [4]. The best way to circumvent this difficulty is to perform the calculation in integer arithmetics. This is convenient if the discrete problem can be cast in a form where the dependent variable assumes only integer values. This is precisely the case for cellular automaton (CA) equations.

The main difficulty that existed till recently for the implementation of this scheme was that one could not produce CA-systems at request and in particular integrable CA's. This obstacle has been bypassed recently thanks to the results of Tokihiro and collaborators [5] who introduced a systematic way to derive a CA from a given discrete equation. To date CA equivalents are known for many of the famous integrable PDE's [6] and also for some ODE's like the Painlevé equations [7]. In the present paper we shall use the CA transcription of a very simple and well-known system, namely the 2-D Lotka-Volterra (LV) model, in order to study the transition from integrability to chaos. The advantage of this system is that it has been thoroughly studied in the continuous case. Its discrete equivalent is also well established and the integrable cases have been identified [8]. In the following sections we shall begin with a brief recall of the continuous and fully discrete results. Then we shall present the ultra-discrete transcription of the LV model, following the procedure of [5]. The study of the ultra-discrete equations of motion will be presented in the next section showing the gradual onset of chaos.

2. Brief recall of the continuous and discrete 2-D Lotka-Volterra models

The continuous 2-D Lotka-Volterra model can be described by the simple differential system:

$$\dot{x} = x(a - y)$$

$$\dot{y} = y(x - b) \tag{2.1}$$

which can easily be written as a second order ODE for one of the variables. We have for instance:

$$\ddot{x} = \frac{\dot{x}^2}{x} + x\dot{x} - b\dot{x} + abx - ax^2 \tag{2.2}$$

Equation (2.1) always has a conserved quantity:

$$K = x + y - a \log y - b \log x \tag{2.3}$$

Thus one could conclude that (2.1) is an integrable system. However, given the form of (2.3), it is clear that (2.1) is not in general *algebraically* integrable, since K is not rational. In fact, if one performs the Painlevé singularity analysis on (2.1), or, equivalently on (2.2), one finds that (2.1) does not possess the Painlevé property, unless:

$$a + b = 0 \tag{2.4}$$

In this case one finds the invariant:

$$x + y = Ce^{at} \tag{2.5}$$

and the integration is reduced to a simple Riccati equation.

The discrete form of the 2-D LV system is also known [8]:

$$\begin{aligned} x_{n+1} - x_n &= \delta(ax_n - x_{n+1}y_{n+1}) \\ y_{n+1} - y_n &= \delta(y_n x_n - by_{n+1}) \end{aligned} \tag{2.6}$$

or, equivalently:

$$\begin{aligned} x_{n+1} &= x_n \frac{1+a}{1+y_{n+1}} \\ y_{n+1} &= y_n \frac{1+x_n}{1+b} \end{aligned} \tag{2.7}$$

where we have taken the discretisation step $\delta = 1$.

The mapping (2.7) is not integrable in general. We can examine its integrability using the criterion known as singularity confinement. It turns out that the integrability condition is:

$$(1 + a)(1 + b) = 1. \tag{2.8}$$

We can rewrite the mapping as:

$$\begin{aligned} x_{n-1} &= \frac{1}{1+a} x_n(1 + y_n) \\ y_{n+1} &= (1 + a)y_n(1 + x_n) \end{aligned} \tag{2.9}$$

The integrability of (2.9) is obtained as a special case of the Gambier mapping (analyzed in [9]). In fact, putting:

$$w_n = (1 + x_n)(1 + y_n) \tag{2.10}$$

we find that (2.9) becomes:

$$x_{n-1} = \frac{w_n x_n}{(1+a)(1+x_n)}$$

$$y_{n+1} = \frac{(1+a)w_n y_n}{1+y_n} \tag{2.11}$$

The auxiliary variable w obeys a linear equation:

$$w_{n+1} = (1+a)w_n - a \tag{2.12}$$

which can be readily integrated to $w_n = 1 + K(1+a)^n$. Once w is given x and y can be obtained through the discrete Riccati (homographic) mappings (2.11).

3. The ultra-discretisation of the 2-D Lotka-Volterra system

The ultra-discrete limit of the 2-D Lotka-Volterra system can be obtained in a straightforward way following the procedure of reference [5]. We put $x = e^{X/\epsilon}$, $y = e^{Y/\epsilon}$, $1 + a = e^{A/\epsilon}$, $1 + b = e^{B/\epsilon}$, and transform (2.7) to:

$$X_{n+1} - X_n = A - \epsilon \log(1 + e^{Y_{n+1}/\epsilon})$$

$$Y_{n+1} - Y_n = \epsilon \log(1 + e^{X_n/\epsilon}) - B \tag{3.1}$$

Next, we take the limit $\epsilon \to 0$ and we use the identity:

$$\lim_{\epsilon \to 0} \epsilon \log(1 + e^{Z/\epsilon}) = \max(0, Z) = (Z + |Z|)/2 \equiv (Z)_+$$

where the latter is the truncated power function defined as $(Z)_+ = 0$ for $Z \leq 0$ and $(Z)_+ = Z$ for $Z \geq 0$. The ultradiscrete LV system is then written as:

$$X_{n+1} - X_n = A - (Y_{n+1})_+$$

$$Y_{n+1} - Y_n = (X_n)_+ - B \tag{3.2}$$

Equation (3.2) defines a cellular automaton. Indeed, if all A, B, X_0, Y_0 are integer (since the truncated power function of an integer yields an integer result) the iteration of (3.2) leads to integer values.

Let us first remark that we can always ensure $B \geq A$. Indeed, if $B < A$, by exchanging X and Y as well as the direction of evolution (towards decreasing n's instead of increasing ones) we can convert the system to one where the values of A and B are exchanged. Thus we can assume $B \geq A$. Moreover, in our analysis we will limit ourselves to cases where the motion was bounded at least for some domain of initial conditions which implies that both A and B are positive. In particular the case $B = -A$ corresponding to the ultra-discretisation of the integrable discrete case was ignored because it leads to unbounded motion. With the assumption of positivity for the parameters A and B we can introduce a further simplification. If we accept *rational* values of X, Y,

B, we can always assume $A = 1$ (through a simple scaling). Although this is not of the upmost importance, it does simplify bookkeeping and it is always simpler to follow a *single* parameter. Thus our further analysis will proceed with $A = 1$ and $B \geq 1$. Given the relative values of X and Y we can distinguish four different determinations of (3.2) in the X, Y plane. They are represented in Figure 1. The corresponding equations of motion are as follows:

$$(I) \qquad X_{n+1} = B + 1 - Y_n \quad \text{and} \quad Y_{n+1} = Y_n + X_n - B$$

$$(II) \qquad X_{n+1} = X_n + B + 1 - Y_n \quad \text{and} \quad Y_{n+1} = Y_n - B$$

$$(III) \qquad X_{n+1} = X_n + 1 \quad \text{and} \quad Y_{n+1} = Y_n - B$$

$$(IV) \qquad X_{n+1} = X_n + 1 \quad \text{and} \quad Y_{n+1} = Y_n + X_n - B$$

Having these explicit forms of the system for each region one can follow the evolution without difficulty. As an illustration, let us show that the motion always possess a stable region delimited by an hexagon. Indeed let us suppose that X and Y are positive or zero. We obtain in this case the equations:

$$Y_{n+1} = Y_n + X_n - B$$

$$X_{n+1} = -Y_n + B + 1 \tag{3.3}$$

If X_0 and Y_0 are given, we can easily see that the iteration of (3.3) leads to any of the 6 points:

$$(X_0, Y_0)$$

$$(-Y_0 + B + 1, X_0 + Y_0 - B)$$

$$(-X_0 - Y_0 + 2B + 1, X_0 - B + 1)$$

$$(-X_0 + 2B, -Y_0 + 2)$$

$$(Y_0 + B - 1, -X_0 - Y_0 + B + 2)$$

$$(X_0 + Y_0 - 1, -X_0 + B + 1)$$

Since we assumed that $X \geq 0$, $Y \geq 0$ and $B \geq 1$ the following inequalities must hold: $B - 1 \leq X_0 \leq B + 1$, $0 \leq Y_0 \leq 2$ and $X_0 + B \leq Y_0 \leq -X_0 + B + 2$. As a result (X_0, Y_0) lies in the hexagonal region with vertices: $(B - 1, 1)$, $(B - 1, 2)$, $(B, 0)$, $(B, 2)$, $(B + 1, 0)$ and $(B + 1, 1)$ that appears in Figures 2,3,4. By construction, this hexagonal region, including its boundary, is stable.

4. A study of the ultra-discrete 2-D Lotka-Volterra system

Let us point out from the outset that most of the results we shall present in this section were obtained by a combination of analytical and symbolic algebra techniques. In fact, the CA character of the system makes the computations with a symbolic algebra program extremely convenient.

We shall not go into all the details of our calculations but limit ourselves to the most pertinent results. For $A = 1$, $B = 1$, the motion is stable if the initial point is chosen in the interior of the heptagon depicted in Figure 2. The internal hexagonal region corresponds to stable orbits of the type we have described at the end of section 3. Between the outermost hexagon and the heptagon one we can get orbits lying on polygons with a number of sides depending on the initial point. In the region outside the heptagon most orbits escape to infinity but some periodic orbits do exist. Similar results can be obtained for the $A = 1$, $B = 2$ case. Here we distinguish three internal stable regions bounded by a hexagon, a heptagon and an octagon (Figure 3). In between these bounding curves the motion is periodic and lies on polygons with various numbers of sides. Again, oustide the octagon we have escape to infinity for most initial conditions, but periodic orbits do exist. Finally, for $A = 1$, $B > 2$ the stable region is characterised by two limiting curves: the always present inner hexagon and an outer heptagon with, in between, stable orbits with various periodicities. In Figure 4 we show these two limiting curves as well as an orbit that lies between these two. Its construction is straightforward once we follow the decomposition of (3.2) into four regions.

Thus the dynamics of the 2D LV system seem particularly simple. One gets either regular bounded or unbounded motion depending on the region one starts from and the borders can be easily obtained. However the situation can be easily perturbed. Let us take $B = 3/2$ (as we have explained rational numbers are easily accomodated in our scheme thanks to the scaling freedom). First, just as in the cases examined above, there exists an inner region where the orbits are periodic and lie on some polygon. This region contains the ever-present inner hexagonal one and is bounded on the exterior by a last periodic orbit corresponding to a polygon with 11 sides. As previously there exists also an outer region where most orbits go to infinity. The interesting feature of the $B = 3/2$ case is that between the two regions of strict periodicity with orbits on polygons and the outer one of unbounded motion there exists a region of more complicated dynamics. Let us examine what happens when we iterate a point with initial coordinates $X_0 = 0$ and for Y_0 successive approximates of the inverse of the golden ratio. Starting from $Y_0 = 2/3$ we find that this point is periodic with period $T = 29$. A higher approximant $Y_0 = 13/21$ gives a period $T = 220$ while $Y_0 = 144/233$ leads to $T = 1165$. Finally the point $Y_0 = 610/987$ gives $T = 11948$. A detailed analysis of this case reveals the existence of two regions separated by an invariant, KAM-like, curve which in this precise case is a 15-sided polygon. The inner of the two regions is the one we just examined. The outer one can be explored starting with points $X_0 = 0$ and Y_0 equal to half the values for the other region. We obtain thus for $Y_0 = 1/3$, $T = 31$, for $Y_0 = 13/42$, $T = 244$, for $Y_0 = 72/233$, $T = 2099$, and for $Y_0 = 305/987$, $T = 8462$. The iterates of these points for the various values of the initial condition Y_0 are given in Figures (5a,b,c,d). In every case the motion is periodic. However, as the denominators of Y_0 increases the period becomes larger and larger. In fact a casual glance at Figures (5a,b,c,d) would suffice to characterize them as chaotic. Thus the mechanism is clear: if one considered the evolution of the mapping in $\mathcal{R}$, then the dynamics would be fully chaotic. Considering successive rational approximations shows how chaos sets in precisely (although the motion is , strictly speaking, periodic all along).

The consideration of other values of B, $1 < B < 2$, does not modify significantly our conclusions. The value of B influences only the number of the “chaotic” regions: up to $B = 3/2$ we have only two of them, while for $B > 3/2$ more “chaotic” regions appear. Still, in all cases studied we have obtained the same pattern: chaotic-looking regions contained between invariant curves with an inner stable region (of strictly periodic motion) and an outer unstable region leading to escape.

5. Conclusion

In this paper we have used the 2-dimensional Lotka-Volterra cellular-automaton in order to study the transtition from integrability to chaos. Our approach is straightforward. We start with equations of motion that can be iterated exactly, provided one works with rational numbers. By increasing the denominator of the parameters, we approximate more and more closely parameters that take values in $\mathcal{R}$. For such values the mapping is not expected to be integrable. What we observe is that the motion, that is strictly periodic at each successive approximation, has longer and longer periods. Moreover, the aspect of the distribution of points becomes progressively more and more intricate and thus one can, based on a visual appreciation, refer to it as chaotic. The advantage of the present approach is that, working with a CA system, we can perform exact computations and thus be certain that our results are not contaminated by unreliable numerics. With the advent of a systematic method for the construction of CA's from a given equation, the method we presented here could lead to a systematic approach for the study of chaos.

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FIGURE CAPTIONS.

Figure 1 : The four evolution domains corresponding to the four different determinations of (3.2) (See text).

Figure 2 : The domain of stability for the $A = 1$, $B = 1$ case.

Figure 3 : The domain of stability for the $A = 1$, $B = 2$ case.

Figure 4 : The domain of stability for the $A = 1$, $B > 2$ case together with the construction of an orbit that lies between the two limiting ones.

Figure 5 : Periodic orbits for $A = 1$, $B = 3/2$ starting from eight different initial points with increasing denominators. For all of them we have $x_0 = 0$ while for y_0 we take a) $y_0 = 2/3$ and $1/3$, b) $y_0 = 26/42$ and $13/42$, c) $y_0 = 144/233$ and $72/233$, d) $y_0 = 610/987$ and $305/987$.

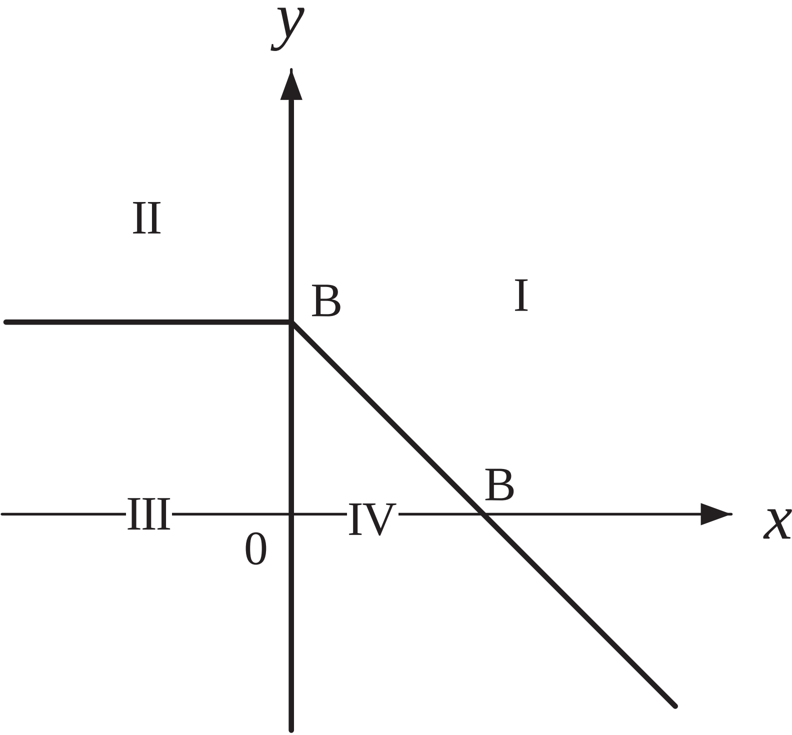


Figure 1

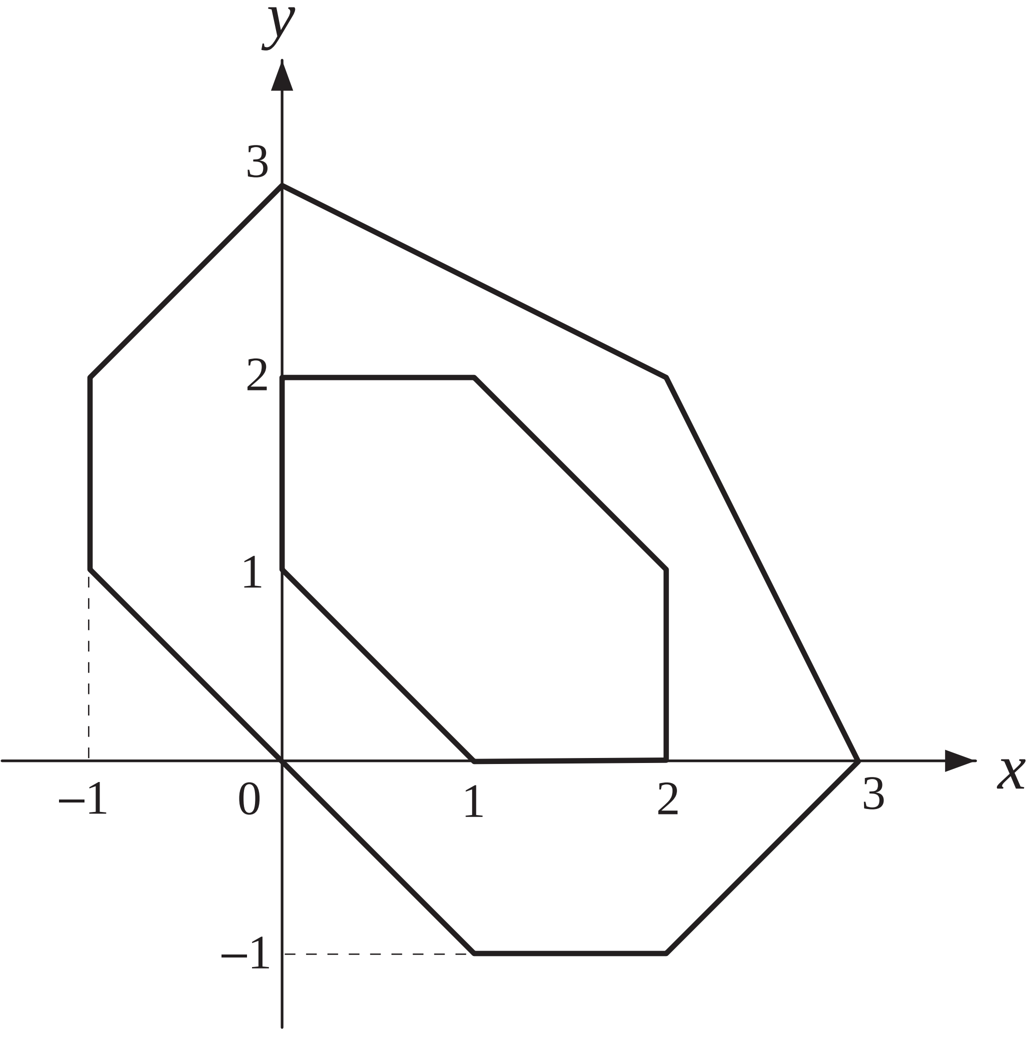
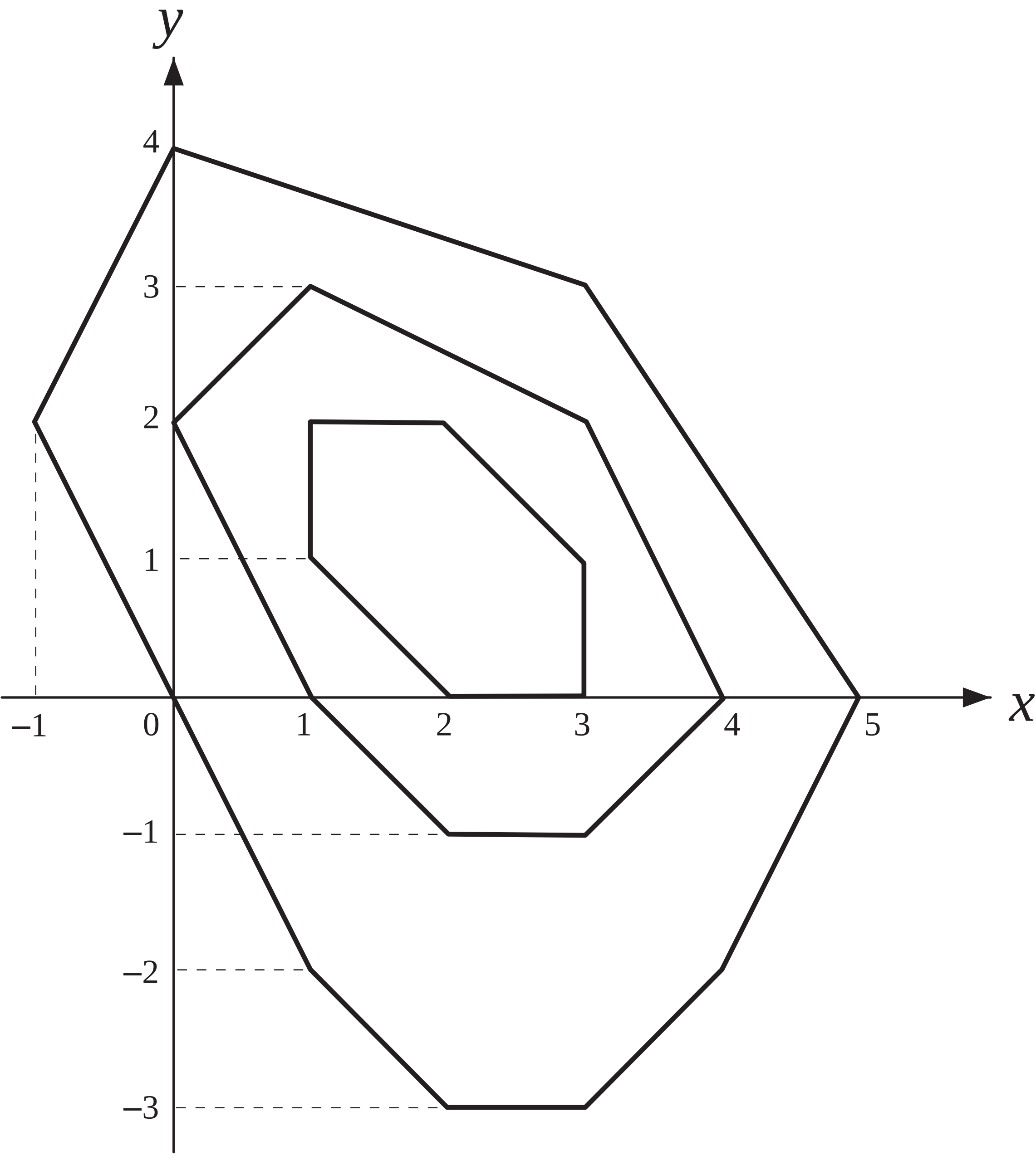


Figure 2



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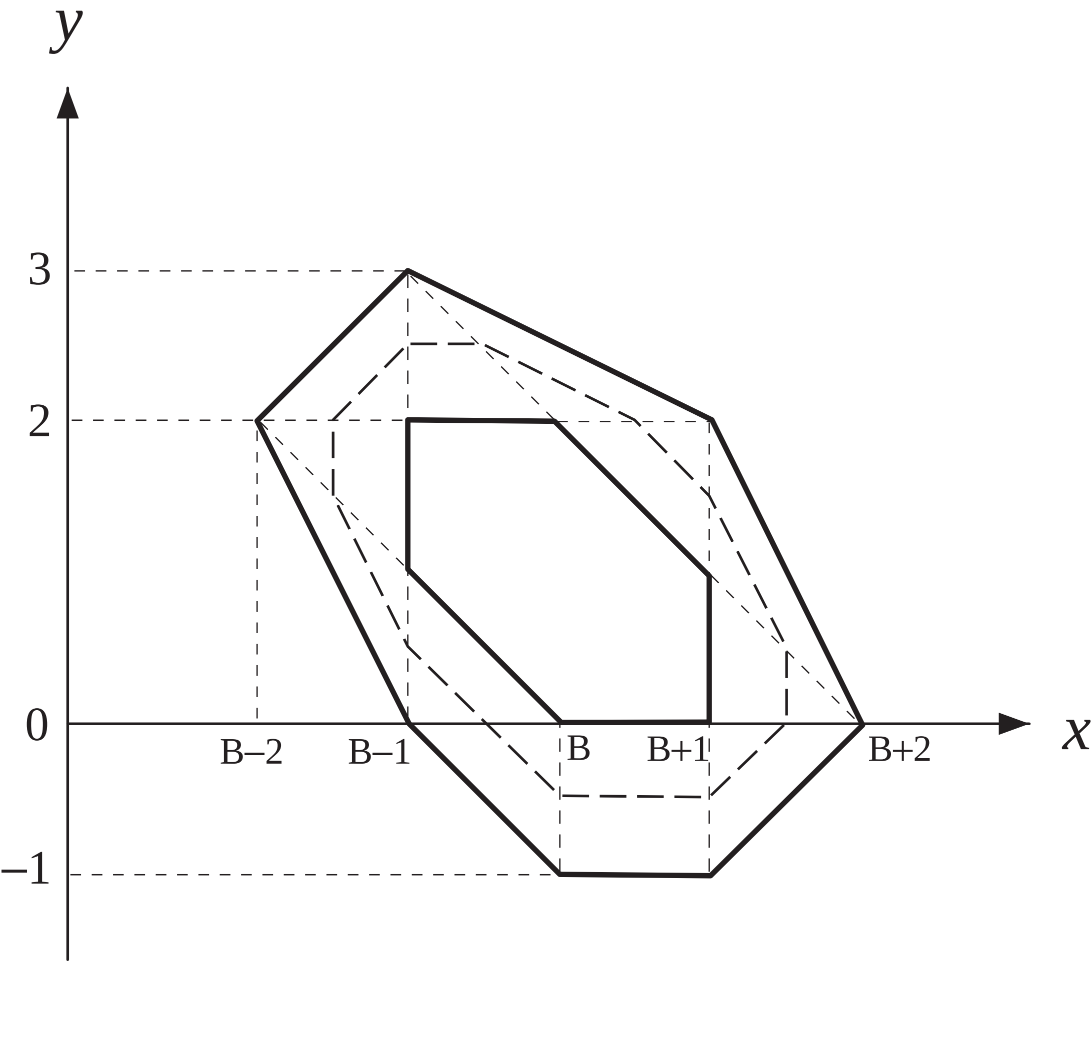

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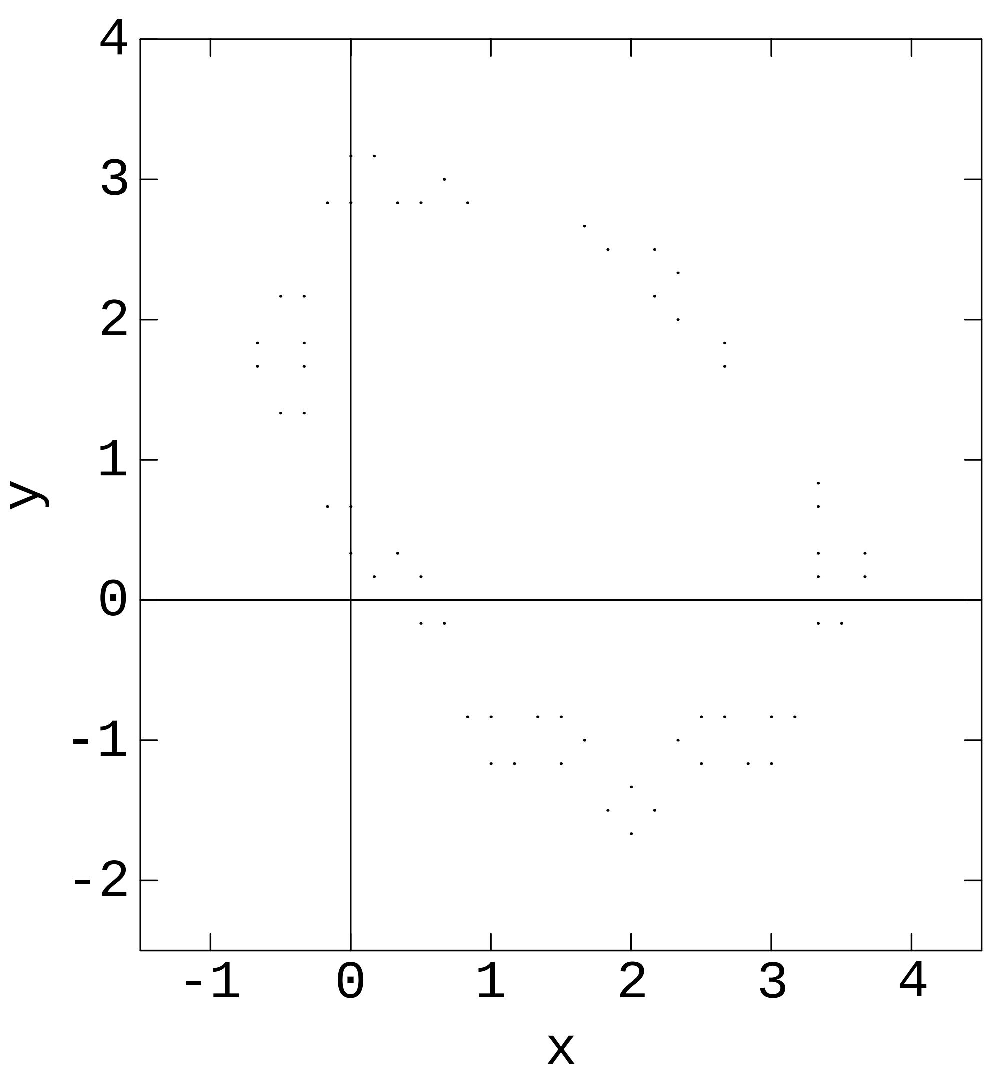

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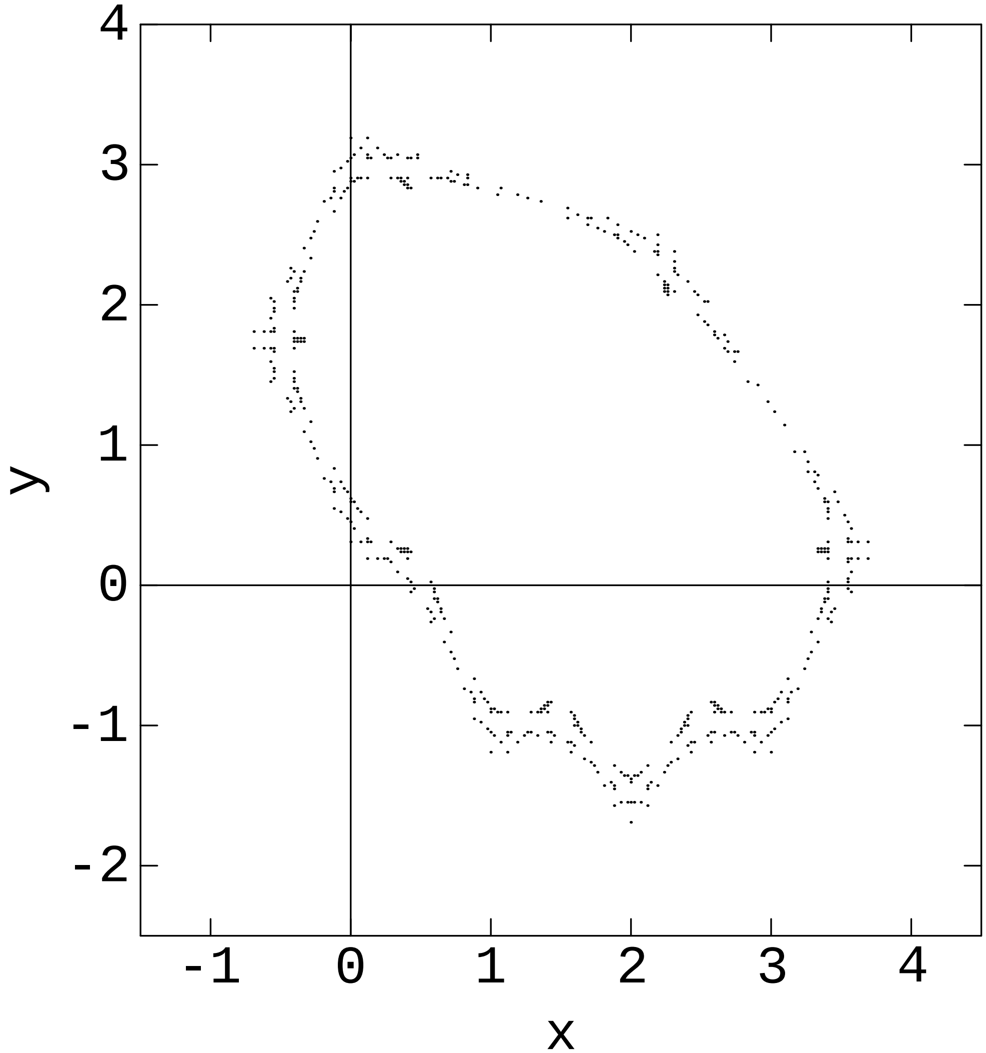

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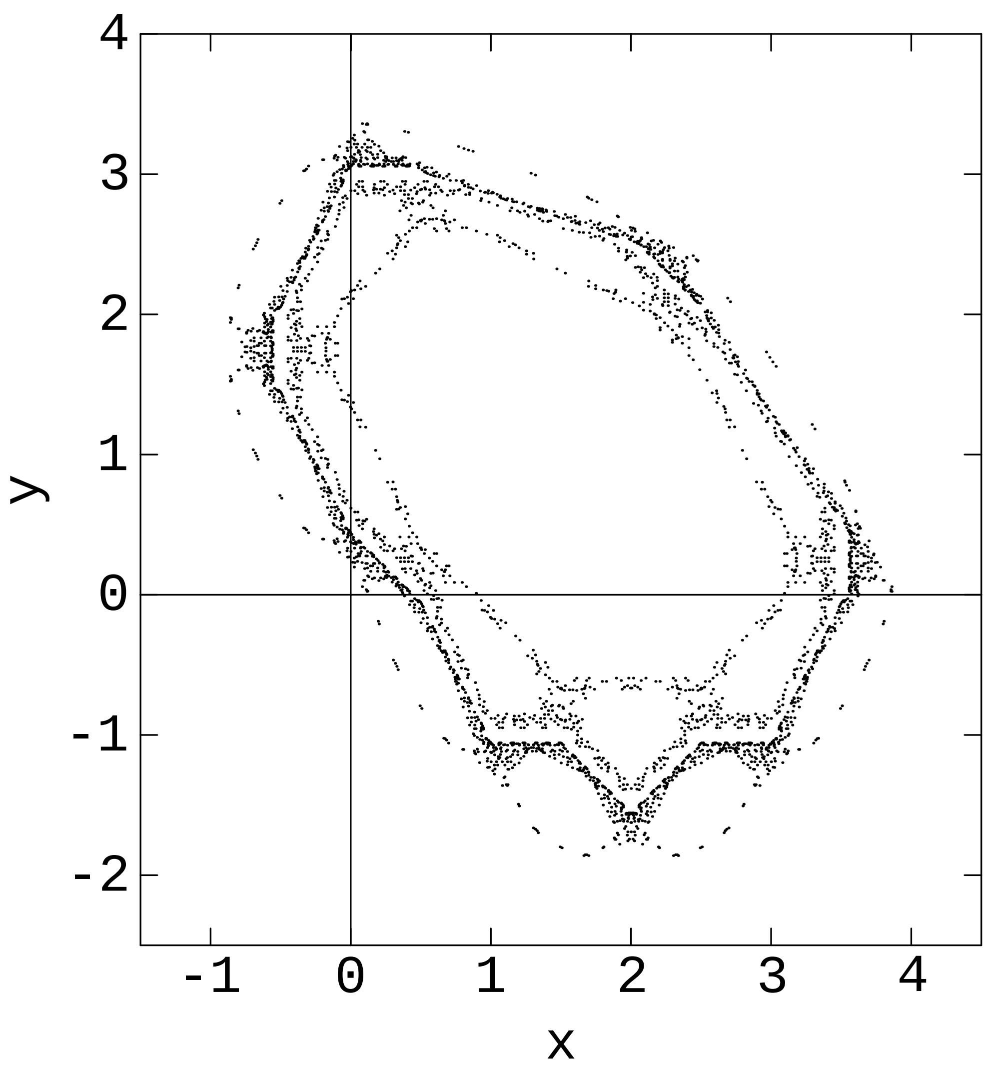

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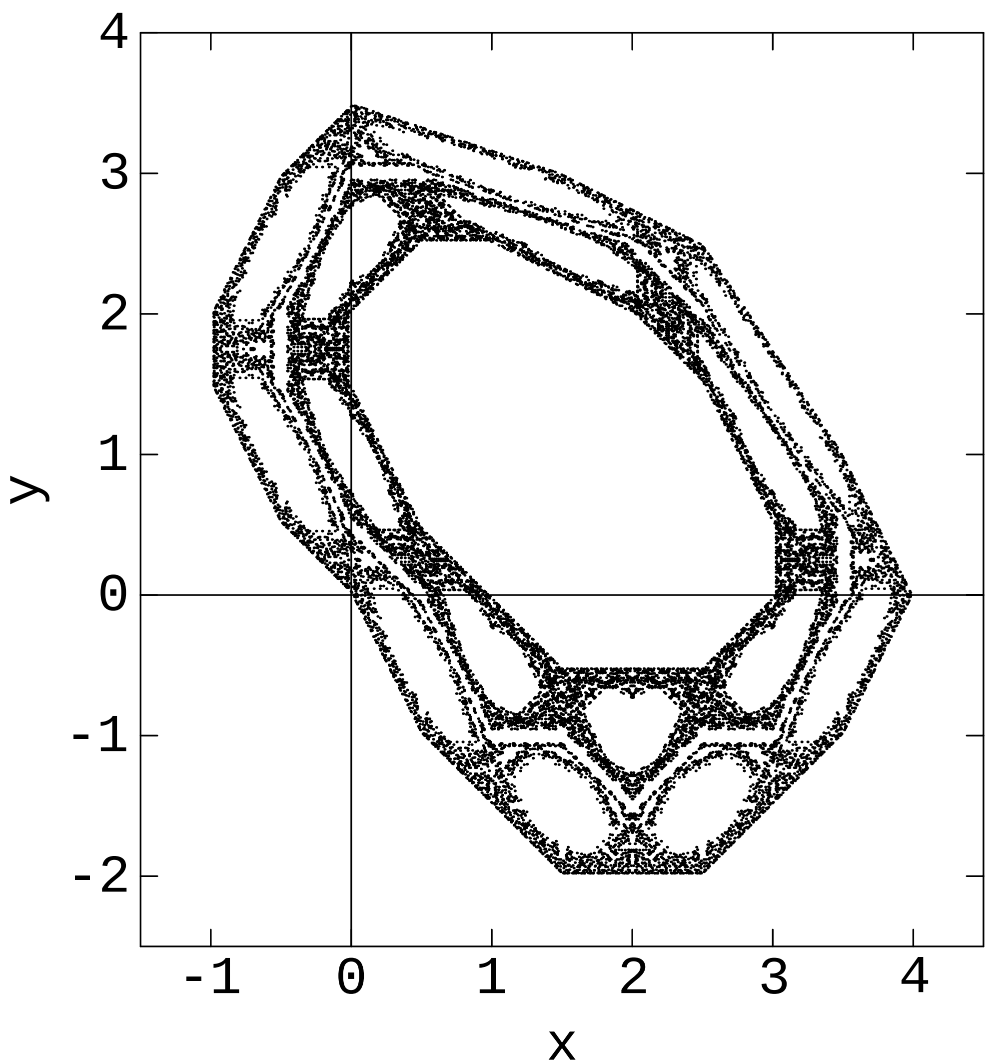

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