Abstracts

International Symposium

Advances in soliton theory and its applications ···The 30th anniversary of the Toda lattice···

December 1 (Sun) - 4 (Wed), 1996

Hayama, Kanagawa, Japan

Yokohama National University

Ultra-discrete Toda Lattice Equation — A Grandchild of Toda —

Daisuke Takahashi
Department of Applied Mathematics and Informatics,
Ryukoku University, Seta, Ohtsu 520-21, Japan

1 Introduction

In 1990, a soliton cellular automaton was discovered by Takahashi and Satsuma[1]. This system is now called 'box and ball system' and is defined by a simple rule using an array of boxes and a finite number of balls. It is a pure soliton system because any state is constructed from solitons and there are an infinite number of conserved quantities[2]. However, its algebraic structure, especially a relation to known continuous soliton systems, was mysterious for a long time. Because all variables are discrete including a dependent one and it was difficult to apply known tools to analyze soliton systems.

The breakthrough came in 1995. In the year, a grandchild of Toda lattice equation now called 'ultra-discrete Toda lattice equation' was discovered[3]. Soon later, a limit process 'ultra-discretization' was discovered and we found that the above box and ball system is equivalent to a ultra-discrete Lotka-Volterra equation[4].

Now, we know how to make the third generation of soliton equations. The new generation has the following features: (1) It is produced from the second generation, that is, fully-discrete soliton equation with discrete independent variables and continuous dependent ones. The relationship of three generations are shown in Fig. 1. (2) Its variables are all discrete including dependent variables. Therefore, the third generation is a cellular automaton (CA). (Here, we use the word 'cellular automaton' in the extended meaning, that is, dependent variable can take any of integer values.)

continuous soliton equation (first generation)
↓
discrete soliton equation (second generation)
↓
ultra-discrete soliton equation (third generation)

Figure 1: Relationship of three generations of soliton equations.

In this article, we will report details of structure of ultra-discrete Toda lattice equation and its relation to the continuous equaiton[5].

2 Ultra-discrete Toda Lattice Equation

In 1967, Toda discovered the Toda lattice equation[6]:

$$\frac{d^2 r_n}{dt^2} = \Delta_n^2 e^{r_n} , \tag{1}$$

where lattice number n is discrete, time t is continuous, and $\Delta_n^2 f_n \equiv f_{n+1} - 2f_n + f_{n-1}$. This equation is the first generation of Toda family. Ten years later, Hirota discovered the discrete Toda lattice equation (d-Toda)[7]:

$$\Delta_t^2 u_n^t = \Delta_n^2 \log(1 + \delta^2(e^{u_n^t} - 1)) , \tag{2}$$

where $\Delta_t^2 f^t \equiv f^{t+1} - 2f^t + f^{t-1}$. This equation is the second generation because all independent variables, n and t are discrete. If we take $u_n^t = r_n(\delta t)$ and $\delta \to 0$, we obtain eq. (1) from eq. (2). Therefore, Toda lattice equation is a limit equation of d-Toda.

Next, we take another specific limit of d-Toda. In this limit, we use a key relation:

$$\lim_{\varepsilon\to+0}\varepsilon\log(e^{A/\varepsilon}+e^{B/\varepsilon}+\cdots)=\max(A,B,\cdots)\ , \quad (3)$$

especially,

$$\lim_{\varepsilon\to+0}\varepsilon\log(1+e^{A/\varepsilon})=\max(0,A)\ . \quad (4)$$

If we set $u_n^t=U_n^t/\varepsilon$ and $\delta=\exp(-L/2\varepsilon)$, we have

$$\Delta_t^2 U_n^t=\Delta_n^2\max(0,U_n^t-L)\ . \quad (5)$$

This equation has a remarkable feature: If initial data (e.g. U_n^0 and U_n^1) and L are all integer, U_n^t for any n and t is always integer. Therefore, this equation is CA because all variables are discrete. We call eq. (5) 'ultra-discrete Toda lattice equation' (u-Toda) and call the limit process from eq. (2) to eq. (5) via the relation (3) 'ultra-discretization'.

Then, a natural question occurs: Is u-Toda a soliton equation, in other words, does u-Toda have N-soliton solution? The answer is yes. We show it by a 2-soliton solution as a typical example. 2-soliton solution of d-Toda is written by

$$u_n^t=\Delta_n^2\log f_n^t\ , \quad (6)$$

and f_n^t is a τ-function of d-Toda,

$$f_n^t=1+e^{\xi_1}+e^{\xi_2}+a_{12}e^{\xi_1+\xi_2}\ , \quad (7)$$

where

$$\xi_i=k_in-\omega_it+\xi_i^0\ , \quad (8)$$

$$\sinh(\omega_i/2)=\sigma_i\delta\sinh(k_i/2)\ , \quad (9)$$

$$\sigma_i=+1 \text{ or } -1\ , \quad (10)$$

$$a_{12}=\frac{\sigma_1\sigma_2-\cosh\frac{1}{2}(k_1+\omega_1-k_2-\omega_2)}{\sigma_1\sigma_2-\cosh\frac{1}{2}(k_1+\omega_1+k_2+\omega_2)} \quad (11)$$

Here k_i and ξ_i^0 are arbitrary parameters. If we introduce new parameters K_i, Ω_i and Ξ_i^0 as

$$k_i=K_i/\varepsilon\ ,\quad \omega_i=\Omega_i/\varepsilon\ ,\quad \xi_i^0=\Xi_i^0/\varepsilon\ , \quad (12)$$

we obtain 2-soliton solution of u-Toda in the limit $\varepsilon\to+0$ as follows:

$$U_n^t=\Delta_n^2\max(0,\Xi_1,\Xi_2,\Xi_1+\Xi_2+A_{12})\ , \quad (13)$$

where

$$\Xi_i=K_in-\Omega_it+\Xi_i^0\ , \quad (14)$$

$$\Omega_i=\sigma_i(\max(0,k_i-L) - \max(0,-k_i-L)) \quad (15)$$

$$\sigma_i=+1 \text{ or } -1\ , \quad (16)$$

$$A_{12}=\begin{cases} -2\min(|K_1|,|K_2|)+L\ , \\ \qquad \text{if } \sigma_1=\sigma_2=-1 \\ \max(\min(K_1+\Omega_1,-K_2-\Omega_2), \\ \qquad \min(-K_1-\Omega_1,K_2+\Omega_2))\ , \\ \qquad \text{otherwise}\end{cases} \quad (17)$$

Two examples of solution are shown in Fig. 2. Figure 2 (a) shows a head-on collision of 2-soliton solution of u-Toda where '.' denotes 0. Figure 2 (b) shows an interaction of 4 solitons. In both figures, non-zero numbers behave as soliton and phase shift occurs after the interaction.

Finally, we show the transition process of solution from d-Toda to u-Toda. To see this, we do not take the limit $\varepsilon\to+0$ after introducing ε into the solution u_n^t. Figure 3 shows 2-soliton solutions U_n^t $(=\varepsilon u_n^t)$ at a fixed time t with the same parameters other than ε. When ε is large, the profile is smooth and values of U_n^t on lattice points n are fractional. However, as ε becomes smaller, pulses become sharper and all values on lattice points go to integer rapidly. Therefore, we can consider that ultra-discretization is not a subtle limit and d-Toda changes to u-Toda smoothly.

```
. . 5 . . . . . 2 1 . . .
. . 1 4 . . . 1 2 . . . .
. . . 2 3 . . 3 . . . . .
. . . . 3 2 2 1 . . . . .
. . . . . 5 3 . . . . . .
. . . . 1 2 4 1 . . . . .
. . . . 3 . . 5 . . . . .
. . . 2 1 . . 1 4 . . . .
. . 1 2 . . . . 2 3 . . .
. . 3 . . . . . . 3 2 . .
```

(a)

```
. . 1 4 . . 2 . . 1 . . . 3 . . . .
. . . 2 3 . 1 1 . 1 . . 2 1 . . . .
. . . . 3 2 . 2 . 1 . 1 2 . . . . .
. . . . . 4 1 1 1 1 . 3 . . . . . .
. . . . . . 5 . 2 1 2 1 . . . . . .
. . . . . . 1 4 1 3 2 . . . . . . .
. . . . . . . 2 5 2 2 . . . . . . .
. . . . . . . 2 3 4 1 1 . . . . . .
. . . . . . 1 2 1 2 3 2 . . . . . .
. . . . . . 3 . 1 . 3 3 1 . . . . .
. . . . . 2 1 . 1 . 1 2 4 . . . . .
. . . . 1 2 . . 1 . . 2 2 3 . . . .
. . . . 3 . . . 1 . . 1 1 3 2 . . .
. . . 2 1 . . . 1 . . . 2 . 4 1 . .
```

(b)

Figure 2: (a) Head-on collision of 2 solitons. (b) Interaction of 4 solitons.

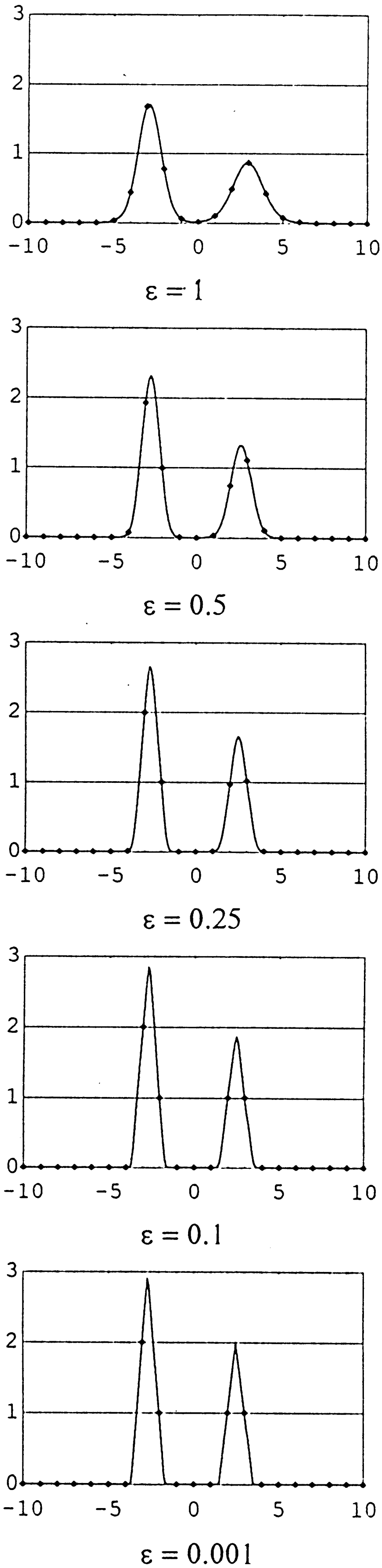

Figure 3: Transition from d-Toda to u-Toda.

Bibliography

[1] D. Takahashi and J. Satsuma: J. Phys. Soc. Jpn. **59** (1990) 3514.

[2] M. Torii, D. Takahashi and J. Satsuma: Physica D **92** (1996) 209.

[3] D. Takahashi and J. Matsukidaira: Phys. Lett. A **209** (1995) 184.

[4] T. Tokihiro, D. Takahashi, J. Matsukidaira and J. Satsuma: Phys. Rev. Lett. **76** (1996) 3247.

[5] J. Matsukidaira, J. Satsuma, D. Takahashi and T. Tokihiro: "Toda-type Cellular Automaton and its N-soliton Solutions", in preparation.

[6] M. Toda: J. Phys. Soc. Jpn. **22** (1967) 431.

[7] R. Hirota: J. Phys. Soc. Jpn. **43** (1977) 2074.