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On some soliton systems defined by using boxes and balls

D. Takahashi

Department of Applied Mathematics and Informatics, Faculty of Science and Technology, Ryukoku University
Seta, Ohtsu, Shiga 520-21, Japan

1. Introduction

The discrete soliton system is a hot theme in the soliton theory. However, few soliton systems of which all dependent and independent variables are discrete have been discovered. In this study, we will propose a family of systems which are fully-discrete soliton systems and some of which can be translated into cellular automata. The features of the systems are as follows:

1. The systems are easily defined by using boxes and balls.
2. Only solitons are generated from any state of the system.
3. Some of the systems have an infinite number of conserved quantities.

First, we define the simplest system.[1] Prepare an infinite number of boxes of the same size. Place them in a row. Then, prepare a finite number of balls of the same size. Assume that each box can hold one ball at most. Put every ball into an arbitrary box. Thus, we obtain a 'state'. A state can be distinguished from the other states by the number of balls and by their positions in the row of boxes. Figure 1 shows an example of a state.

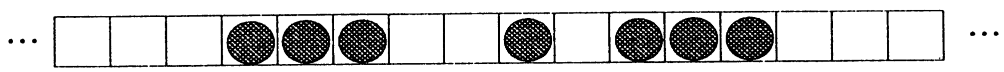

Figure 1: A state of the simplest system.

Next, we define the 'time evolution' of the state. Assume the integer time. The evolution rule of a state from time t to $t+1$ is as follows.

1. All balls move once.
2. The left ball moves earlier.
3. Each ball moves into the nearest empty box on its right.

Figure 2 shows an example of the time evolution of a state. Note that the evolution rule is reversible with respect to time. In this example, 3 ball

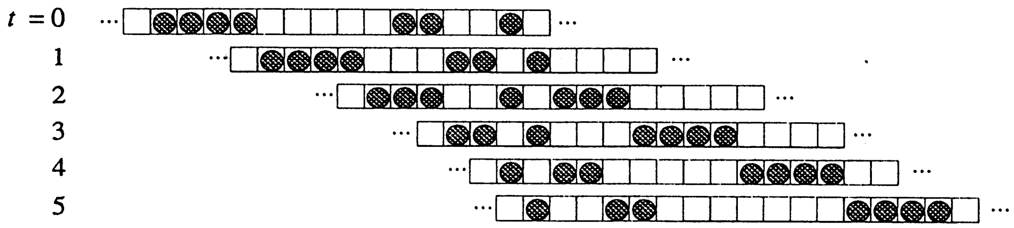

Figure 2: An evolution of a state.

groups exist at $t = 0$ and the numbers of balls of the groups are 4, 2 and 1 from the left. They interact each other at about $t = 2$. At $t = 5$, 3 ball groups again appear. The numbers of balls of the groups are 4, 2 and 1 from the right. Before $t = 0$ and after $t = 5$, all ball groups do not interact each other and move at their own constant speed. Therefore, we can consider that every ball group preserves its identity through the interaction. Generally, a finite number of ball groups preserve their identity through the interaction in the evolution of any state. Thus, the ball group plays a role of a soliton in this system.

If u_j^t denotes the number of balls in the j-th box at time t, $u_j^t = 0$ or 1 and the evolution rule can be expressed by the following equation:

$$u_j^{t+1} = \min\left(1 - u_j^t, \sum_{i=-\infty}^{j-1} (u_i^t - u_i^{t+1})\right) .$$

Therefore, this system can also be a cellular automaton system.

2. Extended Systems

We have found the three types of extension to the simplest system. In all extended systems, the ball groups also preserve their identity in the evolution like solitons. The three types of extension are (1) introducing larger boxes, (2) introducing ball numbers, and, (3) introducing balls which move in the reverse direction. The system (1) was reported in ref. 2, we explain the system (2) and (3) here. First, we define the system (2). Prepare a finite number of balls of the same size. Assume that each ball has a unique number. Then, prepare a row of an infinite number of boxes each of which can hold one ball at most. Put the balls into boxes arbitrarily. Thus, we obtain a state. The evolution rule of the state from time t to $t + 1$ is:

1. All balls move once.

2. The ball with the smaller number moves earlier.

3. Each ball moves into the nearest empty box on its right.

Figure 3 shows an example of the evolution of a state. In this figure, the three ball groups 2457, 16 and 3 exist at $t = 0$. At $t = 6$, the three groups 1356,

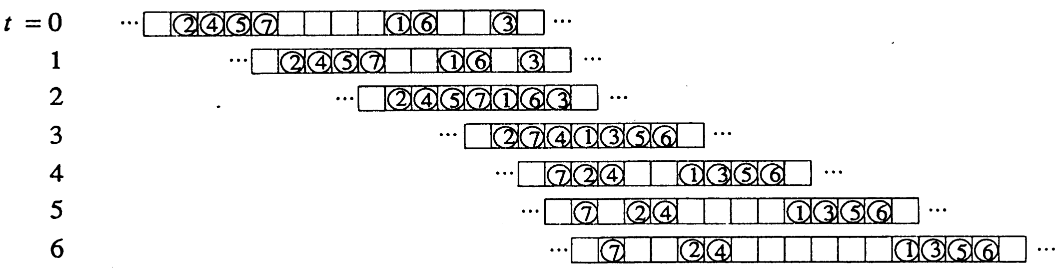

Figure 3: An evolution of a state in the extended system (2).

24 and 7 exist. Assume that the groups 2457, 16 and 3 at $t = 0$ correspond to the groups 1356, 24 and 7 at $t = 6$, respectively. Then, we can consider that three ball groups preserve their own size, that is, the number of balls included, through the interaction though the exchange of balls among the groups occurs. In this system, we have not yet grasp the general dynamics. However, we have verified numerically for the various initial states that the ball groups preserve their own size in the evolution. Therefore, each ball group can be considered to play a role of a soliton in this sense.

Next, we define the system (3). Prepare a finite number of balls with number 1 and those with number 2. Prepare a row of an infinite number of boxes each of which can hold one ball at most. A state is obtained by putting the balls into boxes. The evolution rule of the state from time t to $t+1$ is:

1. First, all balls with number 1 move once. The left one moves earlier. They move to their nearest box on their right which is empty or includes the ball with number 2. If the nearest box is of the latter type, the ball with number 2 is moved to the box in which the ball with number 1 was.

2. Second, all balls with number 2 move once. The right one moves earlier. They move to their nearest empty box on their left.

Figure 4 shows an example of the evolution of a state. In this figure, the

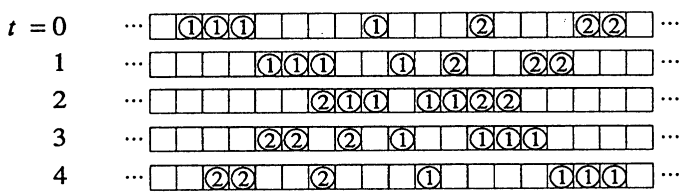

Figure 4: An evolution of a state in the extended system (3).

four ball groups 1111, 1, 2, 22 interact each other and preserve their identity through the interaction. We can prove that all ball groups preserve their identity in the evolution of any state.

Finally, we propose the following conjecture: the ball groups preserve their identity in the hybrid system into which the extension (1) and/or (2) and/or (3) are introduced. This conjecture has been proved partially, and no exception has been found.

REFERENCES

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2. D. Takahashi : *Nonlinear Evolution Equations and Dynamical Systems*, eds. M. Boiti, L. Martina and F. Pempinelli (World Scientific, Singapore, 1992) 245.