NONLINEAR EVOLUTION EQUATIONS AND DYNAMICAL SYSTEMS

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On a Fully Discrete Soliton System

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A system of which dependent variables and independent variables are both discrete is proposed. Any state of this system consists of from boxes and balls and the time evolution of the state is defined by the movement of balls. We call this system '*box and ball system*'. The system has features analogous to those of a continuous soliton system, for example, the Korteweg-de Vries equation [1-4].

I. DEFINITION OF THE BOX AND BALL SYSTEM

In this section, we propose the definition of the box and ball system. Any state of the box and ball system is defined by:

1. There is an array of an infinite number of boxes which holds N balls at most.
2. A finite number of balls exist in some boxes.

Figure 1 shows an example of the state for $N = 3$.

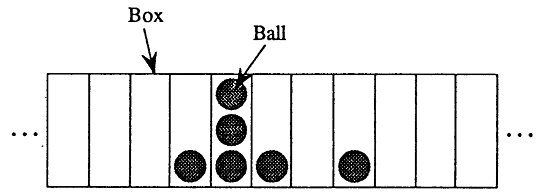


Figure 1 : An example of the state for $N = 3$.

Every ball moves as integer time increases. The rule to determine the state at $t+1$ from that at t is:

1. Every ball moves to its nearest right box which is not full of balls.
2. The left ball moves earlier.
3. All balls move only once.

Figure 2 shows the state at the next time of Figure 1.

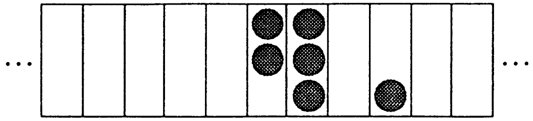
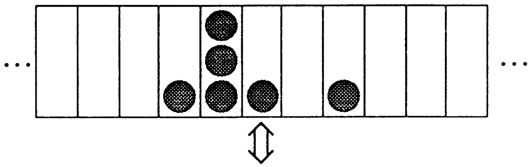


Figure 2 : The state at the next time of Figure 1.

Note that the evolution rule is reversible. The rule to determine the state at $t-1$ from that at t is:

1. Every ball moves to its nearest left box which is not full of balls.
2. The right ball moves earlier.
3. All balls move only once.

If we place the number of balls in each box in a row in place of the picture of box and ball, we have a simple expression of the state. Figure 3 shows the simple expression of the state of Figure 1.



··· 0 0 0 1 3 1 0 1 0 0 0 ···

Figure 3 : The simple expression of the state of Figure 1.

Assume that u_j^t denotes the number of balls in the j-th box at time t. Then, the evolution equation of the state becomes

$$u_j^{t+1} = \min\left(\sum_{i=-\infty}^{j-1}(u_i^t - u_i^{t+1}), N - u_j^t\right) . \qquad (1)$$

If we define S_j^t by $\sum_{i=-\infty}^{j} u_j^t$ and replace $\min(x,y)$ by $\frac{1}{2}\{x+y-|x-y|\}$, (1) reduces to the following finite-difference equation;

$$S_j^{t+1} - S_{j-1}^t = \frac{1}{2}\left\{S_{j-1}^{t+1} - S_j^t + N - \left|S_{j-1}^{t+1} - S_j^t + N\right|\right\} .$$

II. FEATURES OF THE BOX AND BALL SYSTEM

Figure 4 shows the time evolution of the state which is equal to that of Figure 3 at $t = 0$.


```
t = -2   ... 0 0 2 3 0 0 0 1 0 0 ...
    -1         ... 0 0 3 2 0 0 1 0 0 ...          orbit of group
     0            ... 0 0 1 3 1 0 1 0 0 ...
     1               ... 0 0 2 3 0 1 0 0 ...
     2                  ... 0 0 3 2 1 0 0 ...
     3                     ... 0 0 1 2 3 0 0 ...
     4                        ... 0 0 1 0 3 2 0 0 ...
     5                           ... 0 0 1 0 1 3 1 0 0 ...
     6                              ... 0 0 1 0 0 2 3 0 0 ...

                                              group of balls
```

Figure 4 : Interaction of two groups of balls.

In this figure, the following phenomena can be observed:(i) From the state at $t = 0$, a group of 5 balls and a group of 1 ball are produced enough time before and after, respectively. (ii) Each group shifts right at an average speed while both groups are separated far enough from each other. (Orbits of both groups are shown in the figure.) (iii) By the difference of the speed which is due to the number of balls, two groups interact each other at about $t = 2$ and the shift of orbit appears after the interaction. Figure 5 shows the more complicated case for $N = 3$.

```
t = 0   ...0013320003300010200...
    1     ...0001332003300010200...
    2        ...0001330033210200...
    3           ...000330012313200...
    4              ...0033010201332000...
    5                ...0032202000133200 0...
    6                  ...0011313000001332000...
    7                   ...00102033000001332000...
    8                    ...001020033000000001332000...
```

Figure 5 : Interaction of four groups of balls.

This figure shows the interaction of four groups of balls. Generally, we can prove the following fact:

> Any state produces a set of groups enough time before and after, respectively. The set of groups produced before and that produced after are the same, that is, all groups preserve their identities through the interaction. Moreover, the shift of orbit appears after the interaction.

From the above fact, we can consider that each group of balls in this system plays a role of a soliton of a continuous soliton system, and, that the shift of orbit of the group is similar to the phase shift of the soliton.

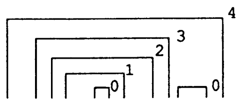
Moreover, we can show that this system has an infinite number of conserved quantities like a continuous soliton system. Though we omit the details here, we explain schematically the procedure to obtain the conserved quantities by using an example. Assume that $N=3$ and a state shown in Figure 6(a) is given. Then, we translate the state into a bit string of infinite length by a certain procedure. Figure 6(b) shows the bit string corresponding to the state of Figure 6(a). Note that every N bits neighboring corresponds to a box and the bit 1 corresponds to a ball.

(a) ...| 0 | 0 | 1 | 3 | 1 | 1 | 0 | 0 | 0 | 0 |...

(b) ...| 000 | 000 | 001 | 111 | 100 | 001 | 000 | 000 | 000 | 000 |...

Figure 6 : (a) A state and (b) the corresponding bit string. Separator '|' is inserted to clarify the correspondence between a box and neighboring three bits.

Next, pair the every bit 1 with a bit 0 on its right so that every pair includes other pairs or nothing. It is clear that all pairs are uniquely determined. Then, the depth of each pair is defined as follows:(i) If a pair includes nothing, its depth is 0. (ii) If a pair includes other pairs of which depth is less than or equal to j (≥ 0), its depth is $j+1$. Figure 7 shows the pairs of the bit string of Figure 6(b) and their depths.



...| 000 | 000 | 001 | 111 | 100 | 001 | 000 | 000 | 000 | 000 |...

Figure 7 : Pairs of a bit 1 and a bit 0 and their depths.

The conserved quantity c_j ($j \geq 0$) is defined by the number of pairs of depth j in the bit string. As for the above example, $c_0 = 2$, $c_1 = c_2 = c_3 = c_4 = 1$, $c_5 = c_6 = \cdots = 0$. The conserved quantities do not change through the time evolution.

III. CONCLUDING REMARKS

From the features described in Section II, we can consider the box and ball system to be a fully discrete version of a soliton system. Especially, any state of this system produces only solitons, that is, groups of balls. Therefore, the dynamics of the system may be much simpler than that of the continuous soliton system.

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