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Soliton Phenomena in a Porous Medium

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I. INTRODUCTION

In the mantle of the Earth, rock is partially melted. Melt phase (melt) and solid phase (matrix) convolute each other complicatedly. Since the density of melt is smaller than that of matrix, melt migrates through the solid interstices. The flow becomes like a porous flow because of this phase mixing. Several researchers studied such a flow and proposed model equations. In 1984, Scott and Stevenson [1] proposed a set of equations which describes the evolution of vertical distribution of melt phase and is equivalent to the following dimensionless equation;

$$u_t = [u^n\{(u^{-m}u_t)_x - 1\}]_x \ , \tag{1}$$

where u is the volume fraction of melt (porosity), t is the time, and x is the vertical space coordinate. Parameters n and m denote the dependency of a matrix permeability k and an effective viscosity η characterizing the rate of matrix compaction and distension as $k \propto u^n$ and $\eta \propto u^{-m}$, respectively. The reasonable values of n and m are $2 \sim 5$ and $0 \sim 1$, respectively. We call (1) the magma equation. Scott and Stevenson numerically showed that pulse-like solitary waves interact one another like solitons.

The magma equation appears in another simple physical situation. It is the flow of two kinds of fluid of which densities are different. If a low-density fluid is injected continuously from a tube at a bottom of a tank filled with a high-density fluid, the former migrates through the latter and forms a thin pipe. If the flux of low-density fluid is controlled at the entrance, the shape of pipe becomes hump-like. Scott, Stevenson and Whitehead [2] studied the interaction of the humps experimentally and found that they behave like solitons. A set of equations which describes such a flow is also proposed by them. It reduces to the following dimensionless equation;

$$u_t + \alpha u u_z + \beta(u_t u_{zz} - u u_{zzt}) = 0 \ , \tag{2}$$

where u is the horizontal cross section of the low-density fluid, t is the time, z is the vertical space coordinate and α, β are constants. Equation (2) is equivalent to the magma equation for $n = 2$ and $m = 1$.

In a preceding paper [3], we have shown explicit travelling wave solutions of (1) for some particular choices of the parameters n and m, and discussed the existence of weak solutions with compact support. Moreover, we proposed a modified version of (1), which reduces to the Korteweg-de Vries (KdV) equation by means of a variable transformation.

In this paper, we give a brief summary of the preceding paper and present some further results on the magma equation. A detailed account will be given in a forthcoming paper [4].

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II. EXPLICIT ANALYTICAL TRAVELLING WAVE SOLUTIONS

In this section, we give explicit travelling wave solutions of (1). Here and hereafter, we confine ourselves to the case $n = 3$ and $m = 0$ for simplicity. Namely we consider

$$u_t = [u^3(u_{tx} - 1)]_x \ . \tag{3}$$

Then, substituting $u = u(z)$, $z = x - ct$ into (3) and integrating twice with respect to z, we obtain an ordinary differential equation governing the travelling wave solutions,

$$\frac{c}{2}u_z^2 = -\frac{1}{u^2}(u^3 - Bu^2 + cu - A) \ , \tag{4}$$

where c is the wave velocity, and A, B are integration constants. If we introduce a transformation of the independent variable from z into ζ,

$$\zeta = \int^z u^{-1}\mathrm{d}z \ , \tag{5}$$

then, (4) is reduced to

$$\frac{c}{2}u_\zeta^2 = -(u^3 - Bu^2 + cu - A) \equiv -f(u) \ , \tag{6}$$

the solutions of which are expressed in terms of the Jacobian elliptic functions. Let us assume that $f(u)$ is factorized as

$$f(u) = (u - u_1)(u - u_2)(u - u_3) \ , \quad u_1 \le u_2 \le u_3 \ . \tag{7}$$

Then a regular solution of (6) is given by

$$u = u_2 + (u_3 - u_2)\,\mathrm{cn}^2 p\zeta \ , \tag{8}$$

where $p = \sqrt{(u_3 - u_1)/2c}$ and cn is the Jacobian elliptic function with the modulus $k = \sqrt{(u_3 - u_2)/(u_3 - u_1)}$. The inverse transformation of (5) is given by

$$z = \int^\zeta u\mathrm{d}\zeta \ . \tag{9}$$

Substituting (8) into (9), we get

$$z = u_1\zeta + \sqrt{2c(u_3 - u_1)}\,\mathrm{E}(s;k) \ , \tag{10}$$

where $s = \mathrm{sn}p\zeta$ and $\mathrm{E}(s;k)$ is the elliptic integral of the second kind defined as $\mathrm{E}(x;k) \equiv \int_0^x \sqrt{(1-k^2t^2)/(1-t^2)}\mathrm{d}t$. Equations (8) and (10) give a periodic wave solution for $u_2 > 0$. The period L is calculated as $L = \sqrt{8c/(u_3 - u_1)}\,\{\mathrm{K}(k) + (u_3 - u_1)\mathrm{E}(k)\}$, where $\mathrm{K}(k)$ and $\mathrm{E}(k)$ are the complete elliptic integrals of the first and the second kind, respectively.

If we take a limit of $k \to 1$ or $u_1 \to u_2$ in (8) and (10), we obtain a solitary wave solution

$$u = u_2 + (u_3 - u_2)\,\mathrm{sech}^2 p\zeta \ , \tag{11}$$

$$z = u_2\zeta + \sqrt{2c(u_3 - u_2)}\,\mathrm{tanh}p\zeta \ . \tag{12}$$

These solutions may correspond to the numerical solutions obtained by Scott and Stevenson [1].

There exists another variable transformation by which solutions can be expressed explicitly in terms of the Jacobian elliptic functions. Let us transform z into ζ as

$$\zeta = \int^{z} u^{-3/2} \mathrm{d}z \,. \tag{13}$$

Then (4) is reduced to

$$\frac{c}{2} u_\zeta^2 = -u(u^3 - Bu^2 + cu - A) \,, \tag{14}$$

from which we obtain a singular periodic wave solution

$$u = \frac{u_1 u_3 \mathrm{sn}^2 p\zeta}{u_3 - u_1(1 - \mathrm{sn}^2 p\zeta)} \,, \tag{15}$$

$$z = \frac{u_1\sqrt{u_2}}{p}\Big[\mathrm{F}(s;k) - (1 - \frac{u_3}{u_1})\Big(\mathrm{E}(s;k) + \frac{\sqrt{(1-s^2)(1-k^2s^2)}}{s}\Big)\Big] \,, \tag{16}$$

where $u_1 > 0$, $p = \sqrt{u_2(u_3 - u_1)/2c}$, $s^2 = 1-(1-\mathrm{sn}^2 p\zeta)u_1/u_3$, $k^2 = (u_3-u_2)/(u_3-u_1)$, and F is the elliptic integral of the first kind defined as $\mathrm{F}(x;k) = \int_0^x 1/\sqrt{(1-t^2)(1-k^2t^2)}\mathrm{d}t$. The maximum and the minimum values of the above periodic solution are u_1 and 0, respectively, and the gradient of $u(z)$ is infinite where $u = 0$. The configuration of $u(z)$ is shown in Figure 1.

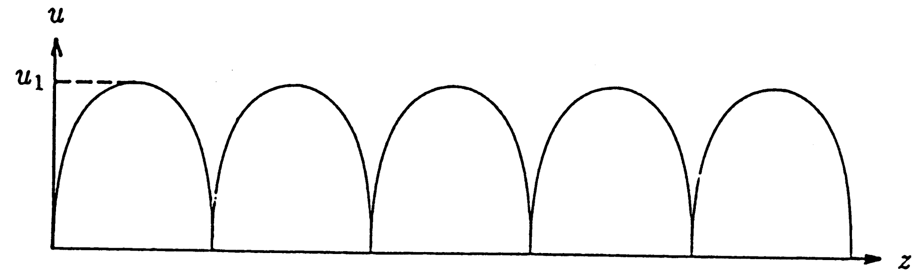

Figure 1. Singular periodic wave solution $u(z)$ described by (15) and (16).

III. COMPACT SUPPORT SOLUTIONS

It is formally possible to cut off one hump of the singular periodic wave given in the preceding section. If we force u to be zero outside the hump, we obtain a weak solution as shown in Figure 2. It satisfies the magma equation except the feet at $z = z_1$ and z_2. We here consider the validity of the existence of such a weak solution.

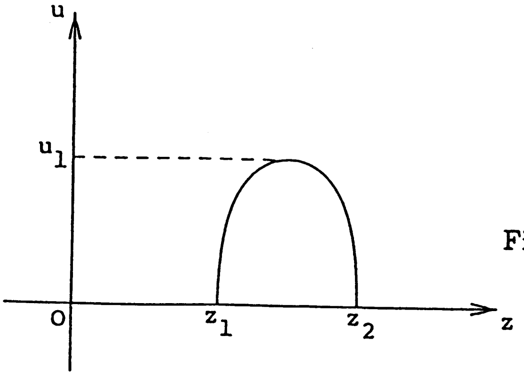

Figure 2. Solitary wave solution with compact support constructed from the solution of Figure 1.

Let us assume that there exists the weak solution as shown in Figure 2, which satisfies (4) on $z_1 < z < z_2$. Then

$$-cu = u^3(-cu_{zz} - 1) - 2A \ , \tag{17}$$

holds on $z_1 < z < z_2$. We note that the integration constant A can not be zero because of the balance of the both sides of (4). On the other hand, in the outside region, $z < z_1$ or $z > z_2$,

$$-cu = u^3(-cu_{zz} - 1) \ , \tag{18}$$

holds because $u \equiv 0$ there. Hence, the solitary wave solution satisfies

$$u_t = [u^3(u_{tx} - 1)]_x + 2A\{\delta(x - ct - z_2) - \delta(x - ct - z_1)\} \ , \tag{19}$$

where $\delta(x)$ is the Dirac's delta function. The last two terms of the right-hand side may be interpreted as source terms for u. Equation (19) equation conserves the total mass, $\int_{-\infty}^{\infty} u dx$. Hence, if the solitary wave solution is applied to (3) which has no source terms, the balance of mass u breaks at the feet of the hump and the solution should change its shape or diverge.

It is suggested from the above argument that solitary wave solutions with compact support are not possible for (3). However, it is probable that such waves exist in real physical systems. We believe that the basic equation should be modified in order to describe them. Still then it is an interesting problem to investigate how the initial data with compact support develops in time for (3).

We here show a result of numerical computation for the initial condition of the single hump given by Figure. 2. The time evolution is calculated with a finite-difference scheme using a potential of u. Figure 3 shows the time development of the profile.

The mass moves to the right more rapidly than the right hand endpoint, and, since the area under the profile must remain constant to conserve mass, the wave steepens as a result of this motion. As the profile steepens and moves to the right, a single solitary wave separates off moving with a constant velocity and profile. This process then repeats as the mass remaining behind the solitary wave(s) splits into another solitary wave (each smaller than its predecessor) and a remainder. At approximately $t = 10$, the profile contains a peak which is, apparently, as steep as can be represented by the numerical grid. Therefore, there exists a possibility that the current results would be of the numerical damping due to the use of finite-difference differentiation. We calculated the same initial value problem with a different number of grid points or with radically different numerical scheme. The results coincide with the above qualitatively. Therefore, it may be suggested that the given numerical results should correspond well to the behavior of a physical system modeled by (3).

The second observation on these results is that the leading solitary wave resembles an analytical solitary wave with very low constant base which is described by (11) and (12). Figure 4 shows a comparison of the leading solitary wave obtained numerically with an analytical solitary wave solution described by (11) and (12); the solid line shows the former and the dotted one shows the latter. Though they are different especially near the x-axis because their origins are quite different, their profiles are very similar to each other as a whole. This argument may be valid for other smaller solitary waves released from the initial hump with compact support. From above discussions, we conclude that an initial data with compact support breaks into some quasi-stable waves which are very similar to solitary waves with low constant base described by (11) and (12), with the help of a damping or a diffusing effect undescribed here.

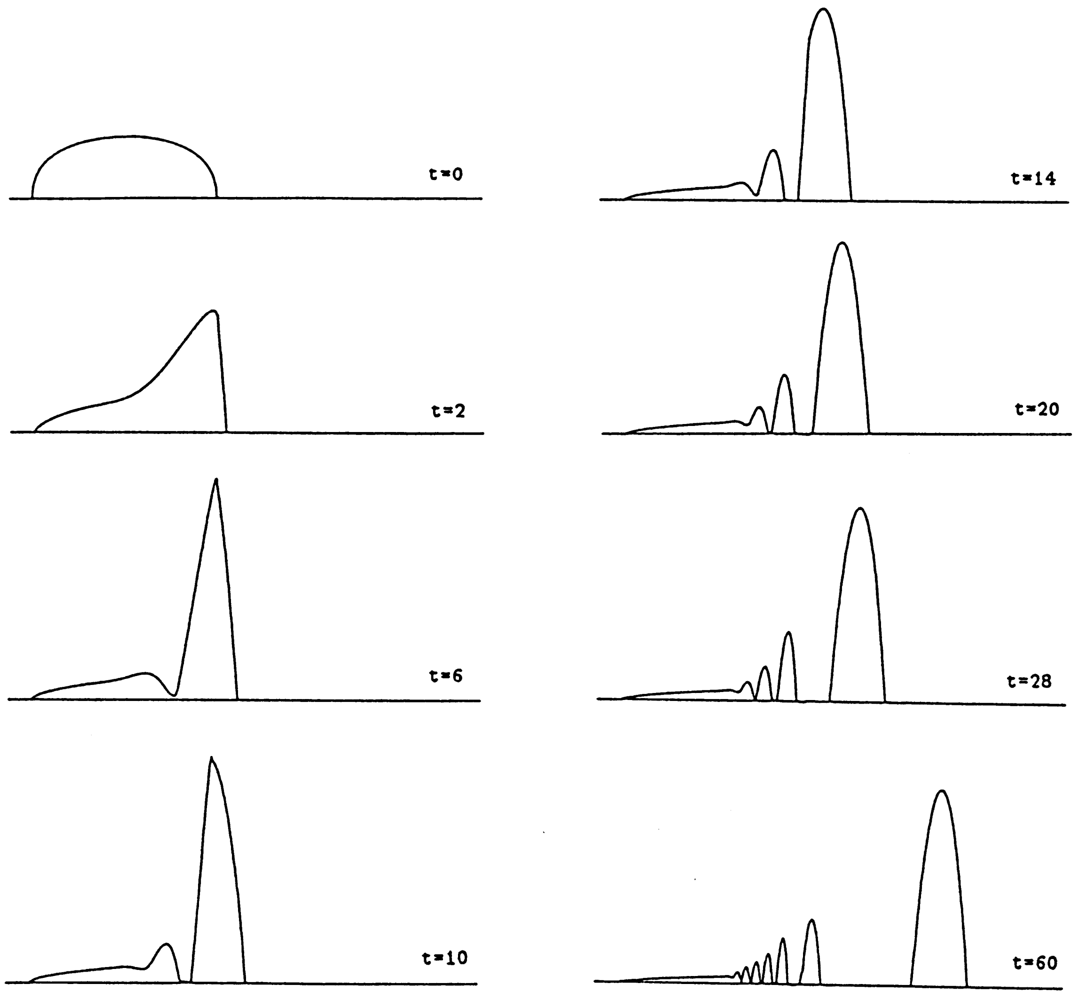

Figure 3. Numerical experiment on the time evolution of the solution of Figure 2.

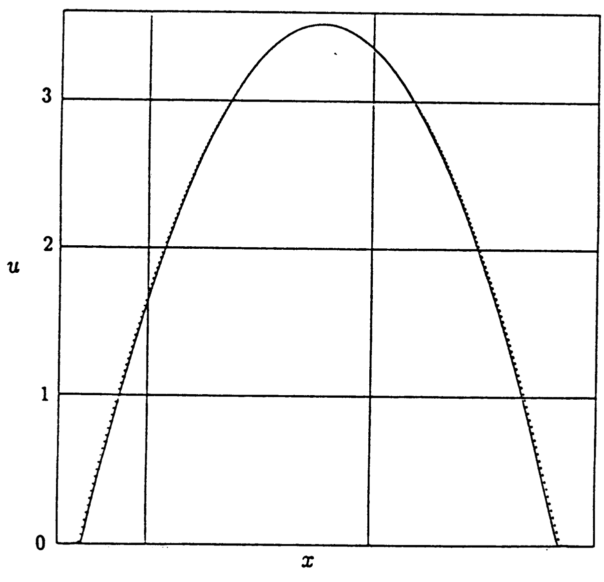

Figure 4. Comparison of the leading solitary wave with an analytical solitary wave described by (11) and (12). The straight line indicates the former and the dotted line the latter.

IV. MODIFIED MAGMA EQUATION

The magma equation has solutions which behave like solitons. Moreover, the results of numerical computation suggests that it may not be integrable. In this section we propose a modified equation which is related to (3) and integrable. The equation is written as

$$u_t + [u^3(3u_{xx} + 1)]_x = 0 \ . \tag{20}$$

We call this the modified magma equation. It is noted that (20) has been introduced by Ito and Kako [5] as an example which possesses higher order conserved quantities. In the long wave and small amplitude approximation, both (3) and (20) have the linear dispersion relation of the type

$$\omega = \alpha k - \beta k^3 \ . \tag{21}$$

If we introduce new independent variables ξ and τ by

$$\xi = \int^x u^{-1} \mathrm{d}x \ , \tag{22}$$

$$\tau = t \ , \tag{23}$$

then (20) reduces to the KdV equation,

$$u_\tau + 6uu_\xi + 3u_{\xi\xi\xi} = 0 \ . \tag{24}$$

By using this fact, we can construct the solitary wave solution with finite constant base of (20),

$$u = c + \frac{3}{2}p^2 \mathrm{sech}^2 \frac{1}{2}(p(\xi + 6c\tau) - 3p^3\tau) \ , \tag{25}$$

$$x = c\xi + 3p\tanh\frac{1}{2}(p(\xi + 6c\tau) - 3p^3\tau) \ , \tag{26}$$

where c is the height of constant base. The solution is quite similar to (11) and (12). Since the KdV equation has N-soliton solution, (20) also has N-soliton solution which describes the interaction of solitary waves of the type (25) and (26).

It is also possible to construct solutions with compact support of (20) from the soliton solutions of the KdV equation. Let us take the 1-soliton solution of (24),

$$u(\xi, \tau) = \frac{3}{2}p^2 \mathrm{sech}^2 \frac{1}{2}(p\xi - 3p^3\tau) \ , \tag{27}$$

where p is an arbitrary parameter. Through the variable transformation (22) and (23), (27) reduces to

$$u(x,t) = \begin{cases} \frac{3}{2}p^2 - \frac{x^2}{6} \ , & |x| \le \sqrt{3}p \ , \\ 0 \ , & |x| > \sqrt{3}p \ , \end{cases} \tag{28}$$

which is a stationary solution with compact support. We note that the incline of u at the feet is finite. If we take the 2-soliton solution of (24) which vanish as $\xi \to \pm\infty$, we get a nonstationary solution with compact support of (20). Figure (5) shows a typical example of time evolution of such a solution.

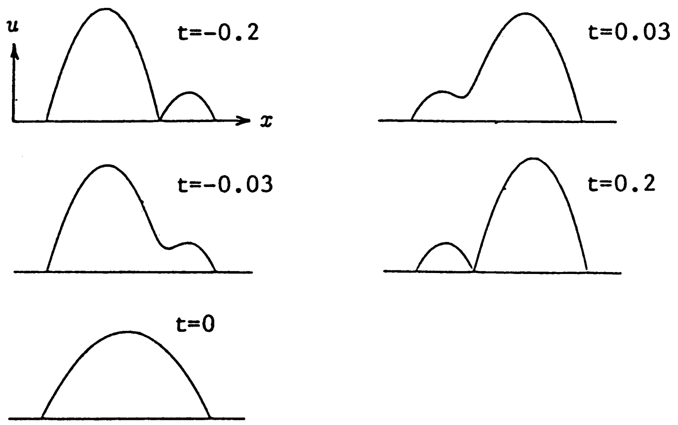

Figure 5. Time evolution of a solution for (20) constructed from the 2-soliton solution of the KdV equation.

Furthermore, if we take the N-soliton solution of (24), we obtain a nonstationary solution describing the interaction of N humps. We note that exact solutions describing this type of interaction of pulses with compact support have been obtained for a nolinear diffusion equation [6].

By using the fact that (20) reduces to the KdV equation through the transformation (22) and (23), we can show the modified magma equation has an infinite number of conserved densities. Let a conserved density of the KdV equation (24) be denoted by I, which satisfies $\frac{\partial}{\partial \tau} \int_{-\infty}^{\infty} I d\xi = 0$. Then, by transforming the variables ξ and τ into x and t, we find that the corresponding conserved density of the modified magma equation (20) is I/u satisfying $\frac{\partial}{\partial t} \int_{x_1}^{x_2} \frac{I}{u} dx = 0$, where x_1 and x_2 are positions of both ends of the positive part of solution. Since one of the conserved densities of the modified magma equation is 1 corresponding to the conserved density u of the KdV equation, we find that the width of compact support is conserved.

Although we do not know any physical relevance of (20) yet, we hope that it plays a role as a clue to the analysis of the systems exhibiting the behavior shown by the above solutions.

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