An Elementary Introduction to Sato Theory

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Abstract

An elementary introduction to Sato theory is given. Starting with an ordinary differential equation, introducing an infinite number of time variables, and imposing a certain time dependence on the solutions, we obtain the Sato equation which governs the time development of the variable coefficients. It is shown that the generalized Lax equation, the Zakharov-Shabat equation and the IST scheme are generated from the Sato equation. It is revealed that the τ-function becomes the key function to express the solutions of the Sato equation. By using the results of the representation theory of groups, it is shown that the τ-function is governed by the partial differential equations in the bilinear forms which are closely related to the Plücker relations.

§1. Introduction

It has been more than two decades since the inverse scattering transform (IST) method was discovered by Gardner, Green, Kruskal and Miura[1)] to solve the initial value problems for the Korteweg-de Vries (KdV) equation,

$$4u_t = 12uu_x + u_{xxx} \ . \tag{1.1}$$

Though the method was at first thought to be applicable to very restricted class of nonlinear wave equations, it has been revealed that it actually applies to a wide class of equations, *i.e.*, the soliton equations. The Lax scheme plays an important role in extending the applicability.[2)]

If we introduce the differential operators,

$$L = \partial^2 + 2u \ , \tag{1.2}$$

$$B = \partial^3 + 3u\partial + \frac{3}{2}u_x \ , \tag{1.3}$$

where ∂ denotes $\partial/\partial x$, then the inverse scattering scheme for the KdV equation may be written by

$$L\psi = \lambda\psi \ , \tag{1.4}$$

$$\frac{\partial\psi}{\partial t} = B\psi \ . \tag{1.5}$$

If the eigenvalue λ is independent of x and t, the compatibility condition of Eqs. (1.4) and (1.5) yields

$$\frac{\partial L}{\partial t} = [B, L] = BL - LB \ , \tag{1.6}$$

which reduces to the KdV equation (1.1). We shall call Eq. (1.6) the Lax equation.

Zakharov and Shabat succeeded in solving the nonlinear Schrödinger equation by extending L to a non-selfadjoint operator.[3)] Ablowitz, Kaup, Newell and Segur gave a unified way of giving IST scheme for various equations including the sine-Gordon equation by means of the expansion in eigenvalues.[4)] We can in principle solve the initial value problems of soliton equations by the IST method.

A formal extension of the IST scheme has also been given by Zakharov and Shabat.[5)] From the compatibility condition between $\partial\psi/\partial t_m = B_m\psi$ and $\partial\psi/\partial t_n = B_n\psi$, we obtain

$$\frac{\partial B_n}{\partial t_m} - \frac{\partial B_m}{\partial t_n} + [B_n, B_m] = 0 \ . \tag{1.7}$$

By choosing suitable differential operators B_n and B_m, we can reduce Eq. (1.7) to several soliton equations. The Kadomtsev-Petviashvili (KP) equation,

$$(4u_t - 12uu_x - u_{xxx})_x - 3u_{yy} = 0 \ , \tag{1.8}$$

is one example of such an equation. We shall call Eq. (1.7) the Zakharov-Shabat equation. In the IST method, N-soliton solutions are obtained by solving the algebraic equations derived from the Gel'fand-Levitan-Marchenko equation. The solutions are expressed in determinant forms.

Hirota's method is another powerful way of obtaining soliton solutions.[6] In this method, we introduce dependent variable transformations to reduce the soliton equations into bilinear forms. Then by using a kind of perturbation technique, N-soliton solutions can be constructed in a systematic way. For example the KP equation is reduced to

$$(4D_xD_t - D_x^4 - 3D_y^2)\,\tau \cdot \tau = 0 \ , \tag{1.9}$$

by the dependent variable transformation,

$$u = (\log \tau)_{xx} \ , \tag{1.10}$$

where the operators D_x etc. are called Hirota's operators and are defined by

$$D_x^n a(x) \cdot b(x) = (\frac{\partial}{\partial x} - \frac{\partial}{\partial x'})^n a(x)b(x')\Big|_{x=x'} = \frac{\partial^n}{\partial s^n} a(x+s)b(x-s)\Big|_{s=0} \ . \tag{1.11}$$

If we apply a formal expansion on τ, we obtain the N-soliton solution in the form of a polynomial in exponential functions. In this method, the transformed dependent variable τ becomes a key function. We shall call it the τ-function hereafter. Hirota's method has been used to obtain not only soliton solutions but several types of special solutions for many nonlinear evolution equations.

There is another way of expressing soliton solutions. It is by means of the Wronski determinant.[7] Freeman and Nimmo have given the Wronskian representation of soliton solutions for various equations.[8]

The diversity of expressing solutions reflects the richness of algebraic structures which the soliton equations possess in common. It is Sato that unveiled the structures by means of the method of algebraic analysis. He noticed that the τ-function of the KP equation is closely related to the Plücker coordinates appearing in the theory of Grassmann manifolds and then discovered that the totality of solutions of the KP equation as well as of its generalization constitutes an infinite dimensional Grassmann manifold.[9] In the concrete, it has been shown that the τ-functions of soliton equations are governed in common by the Plücker relations and can be expressed in terms of determinants with the Wronskian structure.

Shortly after this discovery, Date, Jimbo, Kashiwara and Miwa extended Sato's idea and developed the theory of transformation groups for soliton equations.[10] All these results make it possible to understand the soliton theory from a unified point of view. For example, the relationship among the IST method, Hirota's method and the Bäcklund transformation is clearly explained by the infinite dimensional Lie algebra and its representation on a function space.

A lot of important results have already been presented on these subjects.[11] However, they are rather mathematical and we think it is not so easy to have full understanding of the grand theory. Moreover, the progress after Sato's discovery has been so fast that the detail of original work seems not to be popular even at present. Though there are several lecture notes based on the talks given by Sato[12], they are mostly written in Japanese. There are only two short papers which Sato himself wrote in English.[9],[13]

Considering this situation, we attempt in this paper to explain Sato theory in an elementary fashion. Almost all the results included in the following are due to Sato. Since our aim is to introduce the theory in plain language, we make no attempt to be mathematically rigorous. We hope that this paper may serve as an entrance to this magnificent theory.

In §2, we introduce the microdifferential operator and discuss the relationship between the solutions and the variable coefficients of an ordinary differential equation. In §3, we impose an assumption that the solutions also depend on an infinite number of time variables. By giving the dependency explicitly, we obtain an equation governing the time development of the variable coefficients. We call it the Sato equation, in which the τ-function plays an important role to express the solutions. It is shown in §4 that a generalization of the Lax equation and the Zakharov-Shabat equation can be derived from the Sato equation. In §5 we introduce a linear system associated with the generalized Lax equation. We see that the system reduces to the IST scheme if a certain condition is imposed. The structure of the τ-function is investigated in §6. It has a close relationship to the representation theory of groups. We find that, as a result, the τ-function satisfies partial differential equations which relate to the Plücker relations. The KP equation in its bilinear form comes up as the simplest nontrivial one of such differential equations. In §7 we explicitly express the solutions of the Sato equation in terms of the τ-function. The result also gives the explicit expression of the solutions of the generalized Lax equation and the eigenfunction of its associated linear scheme in terms of the τ-function. The concluding remarks are given in §8. Finally in the appendix, we present some results of the representation theory of groups which are useful for discussing the structures of the τ-function.

§2. Basis of Sato theory

Let us introduce a microdifferential operator,

$$W = 1 + w_1\partial^{-1} + w_2\partial^{-2} + w_3\partial^{-3} + \cdots \ , \tag{2.1}$$

where w_j $(j = 1, 2, \ldots)$ are functions of x and ∂^{-n} is formally defined by

$$\partial^{-n} = (\frac{\mathrm{d}}{\mathrm{d}x})^{-n} \ . \tag{2.2}$$

If we employ the Leibniz rule,

$$\partial^n f(x) = \sum_{r=0}^{\infty} \frac{n(n-1)\ldots(n-r+1)}{r!} \frac{\mathrm{d}^r f}{\mathrm{d}x^r} \partial^{n-r} \ , \tag{2.3}$$

then ∂^n can be a well-defined operator even for negative integer n. For example, we have

$$\partial^{-1} f = f\partial^{-1} - f'\partial^{-2} + f''\partial^{-3} - \cdots \ ,$$
$$\partial^{-2} f = f\partial^{-2} - 2f'\partial^{-3} + 3f''\partial^{-4} - \cdots \ ,$$

where the prime denotes the differentiation in x.

For Eq. (2.1), the inverse operator W^{-1} exists and is written by

$$W^{-1} = 1 + v_1\partial^{-1} + v_2\partial^{-2} + v_3\partial^{-3} + \cdots \ , \tag{2.4}$$

where

$$v_1 = -w_1 \ , \tag{2.5a}$$

$$v_2 = -w_2 + w_1^2 \ , \tag{2.5b}$$

$$v_3 = -w_3 + 2w_1w_2 - w_1w_1' - w_1^3 \ , \tag{2.5c}$$

$$\cdots \ .$$

Though the general theory is developed for $W = \sum_{n=0}^{\infty} w_n(x)\partial^{-n}, w_0 = 1$, we confine ourselves to

$$W_m = 1 + w_1\partial^{-1} + w_2\partial^{-2} + \cdots + w_m\partial^{-m} \ , \tag{2.6}$$

for simplicity.[14)] It is noted that the essence of the general theory is still kept in this simplification.

Let us consider the ordinary differential equation,

$$W_m\partial^m f(x) = (\partial^m + w_1\partial^{m-1} + \cdots + w_m)f(x) = 0 \ , \tag{2.7}$$

which has m linearly independent solutions, $f^{(1)}(x)$, $f^{(2)}(x)$, $\ldots$, $f^{(m)}(x)$. We assume that $f^{(j)}(x)$, $j = 1, 2, \ldots , m$, are analytic, *i.e.*,

$$f^{(j)}(x) = \xi_0^{(j)} + \frac{1}{1!}\xi_1^{(j)}x + \frac{1}{2!}\xi_2^{(j)}x^2 + \cdots, \qquad j = 1, 2, \ldots, m \ . \tag{2.8}$$

Then the rank of the $\infty \times m$ matrix,

$$\Xi = \begin{pmatrix} \xi_0^{(1)} & \xi_0^{(2)} & \cdots & \xi_0^{(m)} \\ \xi_1^{(1)} & \xi_1^{(2)} & \cdots & \xi_1^{(m)} \\ \vdots & \vdots & \cdots & \vdots \end{pmatrix} \ , \tag{2.9}$$

is m and Ξ satisfies

$$W_m\partial^m(1, \frac{x}{1!}, \frac{x^2}{2!}, \ldots)\Xi = 0 \ . \tag{2.10}$$

For an $m \times m$ regular constant matrix R, ΞR also satisfies Eq. (2.10), *i.e.*, Ξ is only unique up to a multiplicative factor in GL$(m, \mathbf{C})$. Hence Ξ can be regarded as an element of

$$\{\infty \times m \text{ matrix of rank } m\}/\mathrm{GL}(m, \mathbf{C}) \ ,$$

which is called the Grassmann manifold GM(m, ∞).

We introduce a shift operator,

$$\Lambda = \begin{pmatrix} 0 & 1 & & & \\ & 0 & 1 & & \mathbf{0} \\ & & 0 & 1 & \\ & & & \ddots & \ddots \\ & \mathbf{0} & & & \ddots \end{pmatrix} \ . \tag{2.11}$$

Then, we have

$$\exp(x\Lambda) = I + x\Lambda + \frac{x^2}{2!}\Lambda^2 + \cdots$$
$$= \begin{pmatrix} 1 & x & x^2/2! & x^3/3! & \dots \\ & 1 & x & x^2/2! & \dots \\ & & 1 & x & \dots \\ & \mathbf{0} & & 1 & \dots \\ & & & & \ddots \end{pmatrix} . \tag{2.12}$$

Moreover, we find

$$H(x) \equiv \exp(x\Lambda)\,\Xi = \begin{pmatrix} f^{(1)} & f^{(2)} & \dots & f^{(m)} \\ \partial f^{(1)} & \partial f^{(2)} & \dots & \partial f^{(m)} \\ \partial^2 f^{(1)} & \partial^2 f^{(2)} & \dots & \partial^2 f^{(m)} \\ \vdots & \vdots & & \vdots \\ \vdots & \vdots & \dots & \vdots \\ \vdots & \vdots & & \vdots \end{pmatrix} . \tag{2.13}$$

We now consider the problem to determine W_m from the solutions, $f^{(1)}, f^{(2)}, \dots, f^{(m)}$. Equation (2.7) yields

$$(\partial^{m-1} f^{(1)})w_1 + (\partial^{m-2} f^{(1)})w_2 + \cdots + f^{(1)}w_m = -\partial^m f^{(1)} ,$$
$$\dots$$
$$(\partial^{m-1} f^{(m)})w_1 + (\partial^{m-2} f^{(m)})w_2 + \cdots + f^{(m)}w_m = -\partial^m f^{(m)} .$$

By solving these simultaneous equations, we find

$$w_j = \frac{\begin{vmatrix} \partial^{m-1} f^{(1)} & \dots & -\partial^m f^{(1)} & \dots & f^{(1)} \\ \vdots & \dots & \vdots & \dots & \vdots \\ \partial^{m-1} f^{(m)} & \dots & -\partial^m f^{(m)} & \dots & f^{(m)} \end{vmatrix}}{\begin{vmatrix} \partial^{m-1} f^{(1)} & \dots & \partial^{m-j} f^{(1)} & \dots & f^{(1)} \\ \vdots & \dots & \vdots & \dots & \vdots \\ \partial^{m-1} f^{(m)} & \dots & \partial^{m-j} f^{(m)} & \dots & f^{(m)} \end{vmatrix}} , \tag{2.14}$$

or (from Eq. (2.6)),

$$W_m = \frac{\begin{vmatrix} f^{(1)} & \dots & f^{(m)} & \partial^{-m} \\ \vdots & \dots & \vdots & \vdots \\ \partial^{m-1} f^{(1)} & \dots & \partial^{m-1} f^{(m)} & \partial^{-1} \\ \partial^m f^{(1)} & \dots & \partial^m f^{(m)} & 1 \end{vmatrix}}{\begin{vmatrix} f^{(1)} & \dots & f^{(m)} \\ \vdots & \dots & \vdots \\ \partial^{m-1} f^{(1)} & \dots & \partial^{m-1} f^{(m)} \end{vmatrix}} . \tag{2.15}$$

In Eq. (2.15), the operator ∂^j has to be put in the rightmost position when we evaluate the determinant of the numerator. The denominator of Eq. (2.14) or (2.15) is the determinant of the matrix consisting of the first m rows of $H(x)$, and is a Wronskian.

§3. Sato equation

We here assume that w_j $(j = 1, 2, \ldots)$ in Eq. (2.1) are also the functions of an infinite number of time variables $t = (t_1, t_2, t_3, \ldots)$. Then the solutions of Eq. (2.7), $f^{(j)}$, include t_1, t_2, t_3, $\ldots$ as parameters,

$$f^{(j)} = f^{(j)}(x; t) = f^{(j)}(x; t_1, t_2, \ldots) \ ,$$

and hence H defined by Eq. (2.13) may also be written as $H(x; t)$.

Let us impose the condition that $H(x; t)$ evolves in time as

$$H(x; t) = \exp x\Lambda \, \exp \eta(t, \Lambda) \, \Xi \ , \tag{3.1}$$

where

$$\eta(t, \Lambda) = \sum_{n=1}^{\infty} t_n \Lambda^n \ . \tag{3.2}$$

If we formally expand the operator,

$$\exp\{(x + t_1)\Lambda + t_2\Lambda^2 + t_3\Lambda^3 + \cdots\} = \sum_{n=0}^{\infty} p_n \Lambda^n \ , \tag{3.3}$$

we find

$$p_n(x + t_1, t_2, t_3, \ldots) = \sum_{\substack{\nu_0+\nu_1+2\nu_2+3\nu_3+\cdots=n \\ \nu_0,\nu_1,\nu_2,\nu_3,\cdots\geq 0}} \frac{x^{\nu_0} t_1^{\nu_1} t_2^{\nu_2} \cdots}{\nu_0! \nu_1! \nu_2! \cdots} \ . \tag{3.4}$$

The first few p_n's are

$$p_0 = 1 \ , \tag{3.5a}$$

$$p_1 = x + t_1 \ , \tag{3.5b}$$

$$p_2 = \frac{1}{2}(x + t_1)^2 + t_2 \ , \tag{3.5c}$$

$$p_3 = \frac{1}{6}(x + t_1)^3 + (x + t_1)t_2 + t_3 \ . \tag{3.5d}$$

These polynomials have the property

$$\frac{\partial p_n}{\partial t_m} = p_{n-m} , \qquad (p_n = 0 \ \text{ for } \ n < 0) , \tag{3.6}$$

and, especially,

$$\frac{\partial p_n}{\partial x} = p_{n-1} . \tag{3.7}$$

We remark that the p_n's play an important role in the representation theory of groups (see Appendix). By means of these polynomials, $H(x;t)$ is expressed by

$$\begin{aligned} H(x;t) &= \begin{pmatrix} 1 & p_1 & p_2 & p_3 & \cdots \\ & 1 & p_1 & p_2 & \cdots \\ & & 1 & p_1 & \cdots \\ & 0 & & 1 & \cdots \\ & & & & \ddots \end{pmatrix} \begin{pmatrix} \xi_0^{(1)} & \xi_0^{(2)} & \cdots & \xi_0^{(m)} \\ \xi_1^{(1)} & \xi_1^{(2)} & \cdots & \xi_1^{(m)} \\ \xi_2^{(1)} & \xi_2^{(2)} & \cdots & \xi_2^{(m)} \\ \vdots & \vdots & \cdots & \vdots \end{pmatrix} \\ &\equiv \begin{pmatrix} h_0^{(1)}(x;t) & h_0^{(2)}(x;t) & \cdots & h_0^{(m)}(x;t) \\ h_1^{(1)}(x;t) & h_1^{(2)}(x;t) & \cdots & h_1^{(m)}(x;t) \\ \vdots & \vdots & \cdots & \vdots \end{pmatrix} . \end{aligned} \tag{3.8}$$

The elements of the above matrix, $h_n^{(j)}(x;t)$, have the following properties:

$$h_0^{(j)}(x;0) = f^{(j)}(x) , \tag{3.9}$$

and

$$h_n^{(j)}(x;t) = \frac{\partial h_0^{(j)}(x;t)}{\partial t_n} = \frac{\partial^n h_0^{(j)}(x;t)}{\partial x^n} . \tag{3.10}$$

Namely, $h_0^{(j)}(x;t)$ is the solution of a set of linear partial differential equations,

$$\left(\frac{\partial}{\partial t_n} - \frac{\partial^n}{\partial x^n}\right) h(x;t) = 0 , \qquad n = 1, 2, \ldots , \tag{3.11}$$

with the initial value

$$h(x;0) = f^{(j)}(x) . \tag{3.12}$$

We consider that W_m (and w_j) in Eq. (2.6) also depend on t so that

$$\begin{aligned} & W_m(x;t)\partial^m h_0^{(j)}(x;t) \\ &\quad = (\partial^m + w_1(x;t)\partial^{m-1} + \cdots + w_m(x;t)) h_0^{(j)}(x;t) \\ &\quad = 0 , \qquad j = 1, 2, \ldots, m . \end{aligned} \tag{3.13}$$

In the similar way to §2, we see that $w_j(x;t)$ and $W_m(x;t)$ are expressed by

$$w_j(x;t) = \frac{\begin{vmatrix} h_{m-1}^{(1)} & \cdots & -h_m^{(1)} & \cdots & h_0^{(1)} \\ \vdots & \cdots & \vdots & \cdots & \vdots \\ h_{m-1}^{(m)} & \cdots & -h_m^{(m)} & \cdots & h_0^{(m)} \end{vmatrix}}{\begin{vmatrix} h_{m-1}^{(1)} & \cdots & h_{m-j}^{(1)} & \cdots & h_0^{(1)} \\ \vdots & \cdots & \vdots & \cdots & \vdots \\ h_{m-1}^{(m)} & \cdots & h_{m-j}^{(m)} & \cdots & h_0^{(m)} \end{vmatrix}} , \tag{3.14}$$

and

$$W_m(x;t) = \frac{\begin{vmatrix} h_0^{(1)} & \cdots & h_0^{(m)} & \partial^{-m} \\ \vdots & \cdots & \vdots & \vdots \\ h_{m-1}^{(1)} & \cdots & h_{m-1}^{(m)} & \partial^{-1} \\ h_m^{(1)} & \cdots & h_m^{(m)} & 1 \end{vmatrix}}{\begin{vmatrix} h_0^{(1)} & \cdots & h_0^{(m)} \\ \vdots & \cdots & \vdots \\ h_{m-1}^{(1)} & \cdots & h_{m-1}^{(m)} \end{vmatrix}} , \tag{3.15}$$

respectively. In §7, we will show that the numerators of $w_j(x;t)$ can be expressed by certain derivatives of the denominator. Hence, the denominator of Eq. (3.14) or (3.15) becomes an important function which yields all the w_j's. We shall also see in §6 that it actually gives the τ-function mentioned in §1.

We now pose a problem to determine the time evolution equation for $W_m(x;t)$. Differentiating (3.13) by t_n and employing the property (3.10), we have

$$(\frac{\partial W_m}{\partial t_n}\partial^m + W_m\partial^m\partial^n)h_0^{(j)}(x;t) = 0 , \tag{3.16}$$

which is the ordinary differential equation of $(m+n)$-th order and shares the same linearly independent solutions as Eq. (3.13). Therefore the differential operators in Eq. (3.16) has to be factorized as

$$\frac{\partial W_m}{\partial t_n}\partial^m + W_m\partial^m\partial^n = B_n W_m \partial^m , \tag{3.17}$$

where B_n is a differential operator. By applying $\partial^{-m}{W_m}^{-1}$ from the right, we have

$$B_n = \frac{\partial W_m}{\partial t_n}{W_m}^{-1} + W_m\partial^n W_m^{-1} . \tag{3.18}$$

From the properties of the operators W_m and ${W_m}^{-1}$, we know that the first term of the right hand side of Eq. (3.18) consists only of the terms with $\partial^{-n}(n = 1, 2, \ldots)$. Hence we have

$$B_n = (W_m\partial^n {W_m}^{-1})^+ , \tag{3.19}$$

where $(\quad)^+$ denotes the differential part of the operator. Consequently, it is shown that the time evolution of $W_m(x;t)$ is governed by

$$\frac{\partial W}{\partial t_n} = B_n W - W\partial^n \ , \tag{3.20}$$

$$B_n = (W\partial^n W^{-1})^+ \ , \tag{3.21}$$

which we call the Sato equation hereafter. The first few of B_n are explicitly given by

$$B_1 = \partial \ , \tag{3.22a}$$

$$B_2 = \partial^2 - 2w_{1,x} \ , \tag{3.22b}$$

$$B_3 = \partial^3 - 3w_{1,x}\partial - 3w_{2,x} + 3w_1 w_{1,x} - 3w_{1,xx} \ , \tag{3.22c}$$

$$\begin{aligned} B_4 = \partial^4 &- 4w_{1,x}\partial^2 - (4w_{2,x} + 6w_{1,xx} - 4w_1 w_{1,x})\partial \\ &- 4w_{3,x} - 6w_{2,xx} - 4w_{1,xxx} + 4w_1 w_{2,x} + 4w_{1,x}w_2 \\ &+ 6w_1 w_{1,xx} + 8{w_{1,x}}^2 - 4w_1^2 w_{1,x} \ , \end{aligned} \tag{3.22d}$$

where $w_{j,\underbrace{x\ldots x}_{\ell}}$ denotes $\partial^\ell w_j/\partial x^\ell$.

Since the similar argument is possible for the operator W defined by Eq. (2.1), we have dropped the suffix m in Eqs. (3.20) and (3.21). It is an important fact in the following discussion that the solutions of Eqs. (3.20) and (3.21) are given in the form of Eq. (3.15). From the results in this section, we see that t_1 plays the same role as x. Hence, we will use t_1 or x without distinction hereafter.

§4. Generalized Lax equation

We have shown in the preceding section that the time evolution of $W(x;t)$ is governed by the Sato equation (3.20) and (3.21), and the coefficients in $W(x;t)$ are expressed in terms of the set of linearly independent solutions of the ordinary differential equations (3.13). In this section we show that a generalization of the Lax equation (1.6) and the Zakharov-Shabat equation (1.7) can be derived from the Sato equation.

We define an operator L by

$$L = W\partial W^{-1} \ , \tag{4.1}$$

which can be written as

$$L = \partial + u_2\partial^{-1} + u_3\partial^{-2} + u_4\partial^{-3} + \cdots \ . \tag{4.2}$$

Substituting Eqs. (2.1) and (2.4) into Eq. (4.1), we find that the coefficients u_j in Eq. (4.2) are related to w_j as

$$u_2 = -w_{1,x} \ , \tag{4.3a}$$
$$u_3 = -w_{2,x} + w_1 w_{1,x} \ , \tag{4.3b}$$
$$u_4 = -w_{3,x} + w_1 w_{2,x} + w_{1,x} w_2 - {w_1}^2 w_{1,x} - {w_{1,x}}^2 \ , \tag{4.3c}$$
$$\cdots .$$

Differentiating L by t_n, and using Eq. (3.20), we have

$$\begin{aligned}\frac{\partial L}{\partial t_n} &= \frac{\partial W}{\partial t_n}\partial W^{-1} + W\partial\frac{\partial W^{-1}}{\partial t_n} \\ &= (B_n W - W\partial^n)\partial W^{-1} - W\partial W^{-1}(B_n W - W\partial^n)W^{-1} \\ &= B_n W\partial W^{-1} - W\partial W^{-1} B_n \ ,\end{aligned} \tag{4.4}$$

where we have used the fact that

$$\frac{\partial(WW^{-1})}{\partial t_n} = W\frac{\partial W^{-1}}{\partial t_n} + \frac{\partial W}{\partial t_n}W^{-1} = 0 \ . \tag{4.5}$$

Consequently, we obtain

$$\frac{\partial L}{\partial t_n} = [B_n, L] = B_n L - L B_n \ , \tag{4.6}$$

which may be called a generalized Lax equation.

It is clear from the definition of L that

$$L^n = W\partial^n W^{-1} \ , \tag{4.7}$$

and therefore B_n may be written by

$$B_n = (L^n)^+ \ . \tag{4.8}$$

If we use u_j instead of w_j, Eqs. (3.22a)$\sim$(3.22d) are expressed as

$$B_1 = \partial \ , \tag{4.9a}$$
$$B_2 = \partial^2 + 2u_2 \ , \tag{4.9b}$$
$$B_3 = \partial^3 + 3u_2\partial + 3u_3 + 3u_{2,x} \ , \tag{4.9c}$$
$$B_4 = \partial^4 + 4u_2\partial^2 + (4u_3 + 6u_{2,x})\partial + 4u_4 + 6u_{3,x} + 4u_{2,xx} + 6u_2^2 \ , \tag{4.9d}$$

respectively.

We now show that an infinite number of equations are derived from the generalized Lax equation. For $n = 2$, Eq. (4.6) are written by

$$
\begin{aligned}
&\frac{\partial u_2}{\partial t_2}\partial^{-1} + \frac{\partial u_3}{\partial t_2}\partial^{-2} + \frac{\partial u_4}{\partial t_2}\partial^{-3} + \cdots \\
&= (\partial^2 + 2u_2)(\partial + u_2\partial^{-1} + u_3\partial^{-2} + \cdots) - (\partial + u_2\partial^{-1} + u_3\partial^{-2} + \cdots)(\partial^2 + 2u_2) \\
&= (u_{2,xx} + 2u_{3,x})\partial^{-1} + (u_{3,xx} + 2u_{4,x} + 2u_2u_{2,x})\partial^{-2} \\
&\qquad + (u_{4,xx} + 2u_{5,x} - 2u_2u_{2,xx} + 4u_{2,x}u_3)\partial^{-3} + \cdots \ .
\end{aligned}
$$

By equating the coefficients of $\partial^{-j}, j = 1, 2, \ldots,$ we get

$$\frac{\partial u_2}{\partial t_2} = u_{2,xx} + 2u_{3,x} \ , \tag{4.10a}$$

$$\frac{\partial u_3}{\partial t_2} = u_{3,xx} + 2u_{4,x} + 2u_2u_{2,x} \ , \tag{4.10b}$$

$$\frac{\partial u_4}{\partial t_2} = u_{4,xx} + 2u_{5,x} - 2u_2u_{2,xx} + 4u_{2,x}u_3 \ , \tag{4.10c}$$

$$\cdots \ .$$

Similarly from Eq. (4.6) with $n = 3$, we get

$$\frac{\partial u_2}{\partial t_3} = u_{2,xxx} + 3u_{3,xx} + 3u_{4,x} + 6u_2u_{2,x} \ , \tag{4.11a}$$

$$\frac{\partial u_3}{\partial t_3} = u_{3,xxx} + 3u_{4,xx} + 3u_{5,x} + 6u_2u_{3,x} + 6u_{2,x}u_3 \ , \tag{4.11b}$$

$$
\begin{aligned}
\frac{\partial u_4}{\partial t_3} &= u_{4,xxx} + 3u_{5,xx} + 3u_{6,x} - 3u_2u_{3,xx} - 3u_{2,xx}u_3 \\
&\qquad + 3u_2u_{4,x} + 9u_{2,x}u_4 + 6u_3u_{3,x} \ ,
\end{aligned} \tag{4.11c}
$$

$$\cdots \ .$$

Following this procedure for $n = 4, 5, \ldots,$ we get an infinite set of equations for the dependent variables $u_2, u_3, u_4, \ldots$. If we eliminate u_3, u_4 from Eqs. (4.10a), (4.10b) and (4.11a), we obtain an equation for u_2,

$$\frac{\partial}{\partial x}\left(4\frac{\partial u_2}{\partial t_3} - 12u_2\frac{\partial u_2}{\partial x} - \frac{\partial^3 u_2}{\partial x^3}\right) - 3\frac{\partial^2 u_2}{\partial t_2^2} = 0 \ , \tag{4.12}$$

which is nothing but the KP equation (1.8). Because this equation is the simplest nontrivial one which is obtained by this formalism, we call the infinite set of equations the KP hierarchy.

It is readily shown that

$$\frac{\partial L^m}{\partial t_n} = [B_n, L^m] \ , \tag{4.13}$$

for m, $n = 1, 2, 3, \ldots$. Hence we have

$$\frac{\partial L^m}{\partial t_n} - \frac{\partial L^n}{\partial t_m} = [B_n, L^m] - [B_m, L^n] \ . \tag{4.14}$$

If we write

$$L^n = B_n - {B_n}^c \ , \tag{4.15}$$

then ${B_n}^c$ consists only of the terms with ∂^{-j}, $j > 0$. Substitution of Eq. (4.15) into Eq. (4.14) yields

$$\begin{aligned} \frac{\partial L^m}{\partial t_n} - \frac{\partial L^n}{\partial t_m} &= [L^n + {B_n}^c, L^m] - [B_m, B_n - {B_n}^c] \\ &= [{B_n}^c, B_m - {B_m}^c] - [B_m, B_n - {B_n}^c] \\ &= [B_n, B_m] - [{B_n}^c, {B_m}^c] \ . \end{aligned} \tag{4.16}$$

Taking the differential part of Eq. (4.16), we obtain

$$\frac{\partial B_m}{\partial t_n} - \frac{\partial B_n}{\partial t_m} = [B_n, B_m] \ , \tag{4.17}$$

which is the Zakharov-Shabat equation (1.7). If we choose $m = 2$ and $n = 3$, then we recover the KP equation (4.12) from Eq. (4.17).

§5. Linear system associated with the generalized Lax equation

A linear system associated with the generalized Lax equation may be written as follows:

$$L\psi = \lambda\psi \ , \tag{5.1}$$

$$\frac{\partial \psi}{\partial t_n} = B_n \psi \ , \tag{5.2}$$

$$\frac{\partial \lambda}{\partial t_n} = 0 \ . \tag{5.3}$$

In fact, Eq. (4.6) is derived from the compatibility condition among Eqs. (5.1)$\sim$(5.3). Equation (5.1) is considered as an eigenvalue problem for the microdifferential operator L

and Eq. (5.2) describes the evolution of the eigenfunction for each of the time variables t_1, t_2, ...

In this section, we first investigate the structure of the eigenfunction. From the definition of L, we find that Eq. (5.1) is rewritten as

$$\partial\psi_0 = \lambda\psi_0 \ , \tag{5.4}$$

where

$$\psi_0 = W^{-1}\psi \ . \tag{5.5}$$

Integration of Eq. (5.4) yields

$$\psi_0 = g(t_2, t_3, \ldots; \lambda)\mathrm{e}^{\lambda x} \ , \tag{5.6}$$

or

$$\psi \ = (1 + \frac{w_1}{\lambda} + \frac{w_2}{\lambda^2} + \cdots)g(t_2, t_3, \ldots; \lambda)\mathrm{e}^{\lambda x} \ , \tag{5.7}$$

where g is an arbitrary function of the arguments. We may assume that g is analytic in λ.

On the other hand, by noticing Eq. (4.15), we have from Eq. (5.2) that

$$\frac{\partial\psi}{\partial t_n} = (L^n + {B_n}^c)\psi \ . \tag{5.8}$$

As mentioned in §4, ${B_n}^c$ consists only of the terms with ∂^{-j}, $j > 0$. From the Leibniz rule (2.3), we see that, for $j = 1, 2, \ldots$,

$$L^{-j} = \partial^{-j} - ju_2\partial^{-j-2} + \cdots \ , \tag{5.9}$$

which means that ∂^{-j} can be expressed in terms of L as

$$\partial^{-j} = \sum_{\ell=j}^{\infty} \tilde{v}_{j\ell}L^{-\ell} \ , \tag{5.10}$$

where $\tilde{v}_{j\ell}$'s are suitable functions of t_1, t_2, ... Hence, Eq. (5.8) may be written by

$$\frac{\partial\psi}{\partial t_n} = (L^n + v_{n1}L^{-1} + v_{n2}L^{-2} + \cdots)\psi \ , \tag{5.11}$$

where v_{nj}'s are also suitable functions of t_1, t_2, ...

From Eq. (5.1), we have

$$L^j\psi = \lambda^j\psi \ . \tag{5.12}$$

Substituting Eq. (5.12) into Eq. (5.11), we obtain

$$\frac{\partial \psi}{\partial t_n} = (\lambda^n + \frac{v_{n1}}{\lambda} + \frac{v_{n2}}{\lambda^2} + \cdots)\psi \ , \tag{5.13a}$$

or

$$\frac{\partial}{\partial t_n} \log \psi = \lambda^n + \frac{v_{n1}}{\lambda} + \frac{v_{n2}}{\lambda^2} + \cdots \ . \tag{5.13b}$$

From Eq. (5.13b), we obtain

$$\log \psi = \sum_{j=1}^{\infty} \lambda^j t_j + t_0 + \sum_{j=1}^{\infty} v_j \lambda^{-j} \ , \qquad (t_0; \text{const.}) \ , \tag{5.14}$$

which may be considered as the Laurent expansion of $\log \psi$ at $\lambda = \infty$ and the first term of the right hand side corresponds to its principal part. In Eq. (5.14), v_j's are again suitable functions of t_1, t_2, ... From Eq. (5.14), we have

$$\psi = \exp(\sum_{j=1}^{\infty} v_j \lambda^{-j}) \exp(t_0 + \lambda t_1 + \lambda^2 t_2 + \cdots) \ . \tag{5.15}$$

Expanding $\exp(\sum_{j=1}^{\infty} v_j \lambda^{-j})$ in $1/\lambda$ and imposing the condition that the resultant ψ coincides with Eq. (5.7) at $t_2, t_3, \ldots = 0$, we finally obtain

$$\psi = (1 + \frac{w_1}{\lambda} + \frac{w_2}{\lambda^2} + \cdots) \exp(t_0 + \lambda t_1 + \lambda^2 t_2 + \cdots) \ . \tag{5.16}$$

Consequently, the eigenfunction of the linear system (5.1) and (5.2) can be directly related to w_j's which are included in the solution of the Sato equation, (3.20) and (3.21).

We now show that the linear system (5.1)$\sim$(5.3) reduces to the IST scheme mentioned in §1, if a certain condition is imposed. Let us assume that L^ℓ consists only of its differential part B_n for a certain ℓ,

$$L^\ell = B_\ell \ . \tag{5.17}$$

This condition also implies that

$$L^{k\ell} = B_{k\ell} \ , \qquad k = 1, 2, \ldots \ . \tag{5.18}$$

Then, Eq. (5.1) reduces to

$$B_{k\ell} \psi = \lambda^{k\ell} \psi \ , \qquad k = 1, 2, \ldots \ . \tag{5.19}$$

Moreover, Eq. (5.2) gives

$$\frac{\partial \psi}{\partial t_{k\ell}} = \lambda^{k\ell}\psi \ , \qquad k = 1, 2, \dots \ . \tag{5.20}$$

Comparing Eq. (5.20) with Eq. (5.16), we find

$$\frac{\partial w_j}{\partial t_{k\ell}} = 0 \ , \qquad j, k = 1, 2, \dots \ , \tag{5.21}$$

which means that all of w_j's are independent of t_ℓ, $t_{2\ell}$, $t_{3\ell}$, ... This situation is called ℓ-reduction.

In the case of 2-reduction, Eq. (5.19) with $k = 1$ is written by

$$(\partial^2 + 2u_2)\psi = \lambda^2\psi \ , \tag{5.22}$$

which is the Schrödinger-type eigenvalue problem. From Eq. (5.17), $L^2 = B_2$, all of the coefficients of ∂^{-j}, $j > 0$ in L^2 must be zero. Then we have, for example,

$$u_3 = -\frac{1}{2}u_{2,x} \ , \tag{5.23a}$$

$$u_4 = \frac{1}{4}u_{2,xx} - \frac{1}{2}u_2^2 \ , \tag{5.23b}$$

$$u_5 = -\frac{1}{8}u_{2,xxx} + \frac{3}{2}u_2 u_{2,x} \ . \tag{5.23c}$$

Here, Eq. (5.2) with $n = 3$ gives

$$\frac{\partial \psi}{\partial t_3} = (\partial^3 + 3u_2\partial + \frac{3}{2}u_{2,x})\psi \ . \tag{5.24}$$

Equations (5.22) and (5.24) taken together are essentially the same as the IST scheme for the KdV equation (1.1).

If we consider

$$\frac{\partial \psi}{\partial t_5} = B_5\psi \ , \tag{5.25}$$

instead of Eq. (5.24), we recover the IST scheme for the fifth-order KdV equation of the Lax-type.

In the case of 3-reduction, we have the third-order eigenvalue problem,

$$B_3\psi = (\partial^3 + 3u_2\partial + 3u_3 + 3u_{2,x})\psi = \lambda^3\psi \ . \tag{5.26}$$

If we here consider

$$\frac{\partial \psi}{\partial t_2} = B_2\psi = (\partial^2 + 2u_2)\psi \ , \tag{5.27}$$

as the associated equation for the time development of ψ, then the compatibility condition of Eqs. (5.26) and (5.27) yields the Boussinesq-type equation,

$$\frac{\partial^2 u_2}{\partial t_2^2} = -\frac{1}{3} u_{2,xxxx} - 2(u_2^2)_{xx} \ . \tag{5.28}$$

Finally, in the case of 4-reduction, the eigenvalue problem, $B_4\psi = \lambda^4\psi$, is of the fourth-order. If $\partial\psi/\partial t_3 = B_3\psi$ is considered as the associated equation, then the compatibility condition yields the coupled KdV equation discussed in Ref. 15).

§6. τ-function

In the preceding two sections we have shown that various analytical schemes to treat soliton equations may be derived from the Sato equation (3.20) and (3.21). Furthermore, we have seen that not only the solutions of soliton equations but the eigenfunctions in the IST scheme are directly related to the solutions of the Sato equation. As mentioned in §3, the latter solutions are expressed in terms of the τ-function,

$$\tau(x;t) = \begin{vmatrix} h_0^{(1)} & \ldots & h_0^{(m)} \\ h_1^{(1)} & \ldots & h_1^{(m)} \\ \vdots & \ldots & \vdots \\ h_{m-1}^{(1)} & \ldots & h_{m-1}^{(m)} \end{vmatrix} . \tag{6.1}$$

In this context, we here study the structures of the τ-function in more detail.

Since the τ-function is the determinant of the matrix made of the first m rows of $H(x;t)$, it may be written as

$$\begin{aligned} \tau(t) &= \det \left[\begin{pmatrix} 1 & p_1 & p_2 & p_3 & \cdots \\ & 1 & p_1 & p_2 & \cdots \\ & \mathbf{0} & \ddots & \ddots & \cdots \\ & & & 1 & \cdots \end{pmatrix} \begin{pmatrix} \xi_0^{(1)} & \xi_0^{(2)} & \cdots & \xi_0^{(m)} \\ \xi_1^{(1)} & \xi_1^{(2)} & \cdots & \xi_1^{(m)} \\ \xi_2^{(1)} & \xi_2^{(2)} & \cdots & \xi_2^{(m)} \\ \vdots & \vdots & & \vdots \\ \vdots & \vdots & \cdots & \vdots \\ \vdots & \vdots & & \vdots \end{pmatrix} \right] \\ &= \det \left({\Xi_0}^t \exp \eta(t, \Lambda)\, \Xi \right) , \end{aligned} \tag{6.2}$$

where ${\Xi_0}^t$ is an $m \times \infty$ matrix defined by

$${\Xi_0}^t = \begin{pmatrix} 1 & 0 & 0 & \ldots & & & \\ & 1 & 0 & 0 & \ldots & & \\ & \mathbf{0} & \ddots & \ddots & \ddots & \ldots & \\ & & & 1 & 0 & 0 & \ldots \end{pmatrix} . \tag{6.3}$$

In this expression we have omitted x-dependence because t_1 plays the same role as x. As a consequence, the p_n's in Eq. (6.2) are defined by

$$p_n(t) = \sum_{\nu_1+2\nu_2+3\nu_3+\cdots=n} \frac{t_1^{\nu_1} t_2^{\nu_2} t_3^{\nu_3} \cdots}{\nu_1! \nu_2! \nu_3! \cdots} \,. \tag{6.4}$$

Using the expansion theorem on the determinant of product of matrices, $\tau(t)$ can be expanded as a sum of the products of determinants;

$$\tau(t) = \sum_{0\le \ell_1<\ell_2<\cdots<\ell_m} \begin{vmatrix} p_{\ell_1} & p_{\ell_2} & \cdots & p_{\ell_m} \\ p_{\ell_1-1} & p_{\ell_2-1} & \cdots & p_{\ell_m-1} \\ \vdots & \vdots & \cdots & \vdots \\ p_{\ell_1-m+1} & p_{\ell_2-m+1} & \cdots & p_{\ell_m-m+1} \end{vmatrix} \times \begin{vmatrix} \xi^{(1)}_{\ell_1} & \xi^{(2)}_{\ell_1} & \cdots & \xi^{(m)}_{\ell_1} \\ \xi^{(1)}_{\ell_2} & \xi^{(2)}_{\ell_2} & \cdots & \xi^{(m)}_{\ell_2} \\ \vdots & \vdots & \cdots & \vdots \\ \xi^{(1)}_{\ell_m} & \xi^{(2)}_{\ell_m} & \cdots & \xi^{(m)}_{\ell_m} \end{vmatrix} \,, \tag{6.5}$$

where the summation is taken over all possible combinations of m nonnegative numbers, ($p_n = 0$ for $n < 0$ as in Eq.(3.6)).

For each set of numbers $(\ell_1, \ell_2, \ldots, \ell_m)$, it is possible to define a corresponding diagram. Let us prepare a chain of cells, each of which is numbered in numerical order (Fig. 1). We put Fermi particles in (i) all of the cells numbered less than and equal to $-m$, and (ii) each of the cells numbered $\ell_1 - m + 1$, $\ell_2 - m + 1$, $\ldots$, $\ell_m - m + 1$. For example, if a set of numbers $(2, 3, 5, 7)$ is given, then the cells are occupied by Fermi particles as Fig. 2. If no number is assigned ($m = 0$), the diagram is as Fig. 3, which may be considered to be the vacuum state. This type of diagram is called the Maya diagram after Sato. It is noted that the correspondence between the set of numbers and this diagram is one to one for fixed m.

Figs. 1∼3

There is also the one to one correspondence between a Maya diagram and a Young diagram. Suppose that we have a Maya diagram. If a cell is occupied by a particle in the diagram, assign a vertical line ↑, and if it is empty, assign a horizontal line →, respectively. Then we obtain a connected line. For the vacuum state we have a line with one corner. The diagram surrounded by these two lines gives the corresponding Young diagram. Figure 4 shows the procedure to construct the Young diagram for the Maya diagram of Fig. 2. The vacuum state itself corresponds to the Young diagram ϕ.

Fig. 4

Thus we find the one to one correspondence between a set of numbers and a Young diagram for fixed m *via* a Maya diagram. This correspondence is quite reasonable. In fact, the Young diagram is introduced to classify the irreducible representation of the symmetric group. Also the determinants composed of p_j's in Eq. (6.5) are the Schur functions themselves which appear in symmetric group theory.

Since the correspondence is one to one, we may denote

$$S_Y(t) = \begin{vmatrix} p_{\ell_1} & p_{\ell_2} & \cdots & p_{\ell_m} \\ p_{\ell_1-1} & p_{\ell_2-1} & \cdots & p_{\ell_m-1} \\ \vdots & \vdots & \cdots & \vdots \\ p_{\ell_1-m+1} & p_{\ell_2-m+1} & \cdots & p_{\ell_m-m+1} \end{vmatrix} , \tag{6.6}$$

and

$$\xi_Y = \begin{vmatrix} \xi^{(1)}_{\ell_1} & \xi^{(2)}_{\ell_1} & \cdots & \xi^{(m)}_{\ell_1} \\ \xi^{(1)}_{\ell_2} & \xi^{(2)}_{\ell_2} & \cdots & \xi^{(m)}_{\ell_2} \\ \vdots & \vdots & \cdots & \vdots \\ \xi^{(1)}_{\ell_m} & \xi^{(2)}_{\ell_m} & \cdots & \xi^{(m)}_{\ell_m} \end{vmatrix} , \tag{6.7}$$

where the suffix Y means the Young diagram corresponding to the set of numbers (ℓ_1, ℓ_2, $\ldots$, ℓ_m). It is noted that, although different sets of numbers may correspond to a certain Y if m is not fixed, the right-hand side of Eq. (6.6) gives the same function for those sets. With these notations, Eq. (6.5) may be written by

$$\tau(t) = \sum_{\phi \leq Y \leq m\{} S_Y(t)\,\xi_Y \ , \tag{6.8}$$

where the summation is taken over all Young diagrams which have less than $m+1$ rows.

The case of $m = 2$ gives a simple example of this expansion:

$$
\begin{aligned}
\tau(t) &= \det \left[\begin{pmatrix} p_0 & p_1 & p_2 & \cdots \\ 0 & p_0 & p_1 & \cdots \end{pmatrix} \begin{pmatrix} \xi_0^{(1)} & \xi_0^{(2)} \\ \xi_1^{(1)} & \xi_1^{(2)} \\ \vdots & \vdots \end{pmatrix} \right] \\
&= \begin{vmatrix} p_0 & p_1 \\ 0 & p_0 \end{vmatrix} \begin{vmatrix} \xi_0^{(1)} & \xi_0^{(2)} \\ \xi_1^{(1)} & \xi_1^{(2)} \end{vmatrix} + \begin{vmatrix} p_0 & p_2 \\ 0 & p_1 \end{vmatrix} \begin{vmatrix} \xi_0^{(1)} & \xi_0^{(2)} \\ \xi_2^{(1)} & \xi_2^{(2)} \end{vmatrix} \\
&\quad + \begin{vmatrix} p_0 & p_3 \\ 0 & p_2 \end{vmatrix} \begin{vmatrix} \xi_0^{(1)} & \xi_0^{(2)} \\ \xi_3^{(1)} & \xi_3^{(2)} \end{vmatrix} + \cdots \\
&= S_\phi \xi_\phi + S_{\Box} \xi_{\Box} + S_{\Box\Box} \xi_{\Box\Box} + \cdots \ .
\end{aligned}
\tag{6.9}
$$

As shown in Appendix, it is possible to expand an arbitrary analytic function $f(t)$ in $S_Y(t)$,

$$
f(t) = \sum_Y S_Y(t)\, c_Y \ . \tag{6.10}
$$

The coefficient c_Y is uniquely determined from the orthogonality condition as

$$
c_Y = S_Y(\tilde{\partial}_t) f(t)|_{t=0} \ , \tag{6.11}
$$

where $\tilde{\partial}_t$ is defined by

$$
\tilde{\partial}_t = \left(\frac{\partial}{\partial t_1}, \frac{1}{2}\frac{\partial}{\partial t_2}, \frac{1}{3}\frac{\partial}{\partial t_3}, \ldots\right) \ . \tag{6.12}
$$

Equation (6.8) is also considered to be the expansion of $\tau(t)$ in $S_Y(t)$. Since the expansion (6.10) is unique, this is also unique.

For the coefficients ξ_Y's of τ-function, there exists an important property that they always have to satisfy constraints called the Plücker relations. Let us derive these relations in an elementary way. We first consider the simplest nontrivial case $m = 2$. For the τ-function (6.9), we choose four numbers, k, $\ell_1 < \ell_2 < \ell_3$, and notice a trivial identity,

$$
\begin{vmatrix} \xi_k^{(1)} & \xi_{\ell_1}^{(1)} & \xi_{\ell_2}^{(1)} & \xi_{\ell_3}^{(1)} \\ \xi_k^{(2)} & \xi_{\ell_1}^{(2)} & \xi_{\ell_2}^{(2)} & \xi_{\ell_3}^{(2)} \\ 0 & \xi_{\ell_1}^{(1)} & \xi_{\ell_2}^{(1)} & \xi_{\ell_3}^{(1)} \\ 0 & \xi_{\ell_1}^{(2)} & \xi_{\ell_2}^{(2)} & \xi_{\ell_3}^{(2)} \end{vmatrix} = \begin{vmatrix} \xi_k^{(1)} & 0 & 0 & 0 \\ \xi_k^{(2)} & 0 & 0 & 0 \\ 0 & \xi_{\ell_1}^{(1)} & \xi_{\ell_2}^{(1)} & \xi_{\ell_3}^{(1)} \\ 0 & \xi_{\ell_1}^{(2)} & \xi_{\ell_2}^{(2)} & \xi_{\ell_3}^{(2)} \end{vmatrix} = 0 \ . \tag{6.13}
$$

By applying the Laplace expansion to the first determinant, we obtain

$$
\begin{aligned}
&\begin{vmatrix} \xi_k^{(1)} & \xi_{\ell_1}^{(1)} \\ \xi_k^{(2)} & \xi_{\ell_1}^{(2)} \end{vmatrix} \begin{vmatrix} \xi_{\ell_2}^{(1)} & \xi_{\ell_3}^{(1)} \\ \xi_{\ell_2}^{(2)} & \xi_{\ell_3}^{(2)} \end{vmatrix} - \begin{vmatrix} \xi_k^{(1)} & \xi_{\ell_2}^{(1)} \\ \xi_k^{(2)} & \xi_{\ell_2}^{(2)} \end{vmatrix} \begin{vmatrix} \xi_{\ell_1}^{(1)} & \xi_{\ell_3}^{(1)} \\ \xi_{\ell_1}^{(2)} & \xi_{\ell_3}^{(2)} \end{vmatrix} \\
&\qquad + \begin{vmatrix} \xi_k^{(1)} & \xi_{\ell_3}^{(1)} \\ \xi_k^{(2)} & \xi_{\ell_3}^{(2)} \end{vmatrix} \begin{vmatrix} \xi_{\ell_1}^{(1)} & \xi_{\ell_2}^{(1)} \\ \xi_{\ell_1}^{(2)} & \xi_{\ell_2}^{(2)} \end{vmatrix} = 0 \ ,
\end{aligned}
\tag{6.14}
$$

which gives constraints among ξ_Y's. For example, if we choose $(k, \ell_1, \ell_2, \ell_3) = (0, 1, 2, 3)$, Eq. (6.14) is reduced to

$$\xi_\phi \xi_{\ydiagram{2,2}} - \xi_{\ydiagram{1}} \xi_{\ydiagram{2,1}} + \xi_{\ydiagram{2}} \xi_{\ydiagram{1,1}} = 0 \ . \tag{6.15}$$

Similarly we have

$$\xi_\phi \xi_{\ydiagram{3,2}} - \xi_{\ydiagram{1}} \xi_{\ydiagram{3,1}} + \xi_{\ydiagram{3}} \xi_{\ydiagram{1,1}} = 0 \ , \qquad \text{for } (k, \ell_1, \ell_2, \ell_3) = (0, 1, 2, 4) \ , \tag{6.16a}$$

$$\xi_\phi \xi_{\ydiagram{3,3}} - \xi_{\ydiagram{2}} \xi_{\ydiagram{3,1}} + \xi_{\ydiagram{3}} \xi_{\ydiagram{2,1}} = 0 \ , \qquad \text{for } (k, \ell_1, \ell_2, \ell_3) = (0, 1, 3, 4) \ , \tag{6.16b}$$

$$\xi_{\ydiagram{1}} \xi_{\ydiagram{3,3}} - \xi_{\ydiagram{2}} \xi_{\ydiagram{3,2}} + \xi_{\ydiagram{3}} \xi_{\ydiagram{2,2}} = 0 \ , \qquad \text{for } (k, \ell_1, \ell_2, \ell_3) = (0, 2, 3, 4) \ , \tag{6.16c}$$

$$\xi_{\ydiagram{1,1}} \xi_{\ydiagram{3,3}} - \xi_{\ydiagram{2,1}} \xi_{\ydiagram{3,2}} + \xi_{\ydiagram{3,1}} \xi_{\ydiagram{2,2}} = 0 \ , \qquad \text{for } (k, \ell_1, \ell_2, \ell_3) = (1, 2, 3, 4) \ , \tag{6.16d}$$

respectively.

Generally for an arbitrary m, by choosing $k_1 < k_2 < \ldots < k_{m-1}$ and $\ell_1 < \ell_2 < \ldots < \ell_{m+1}$, and applying the Laplace expansion to

$$\left| \begin{array}{cccc:cccc} \xi^{(1)}_{k_1} & \xi^{(1)}_{k_2} & \ldots & \xi^{(1)}_{k_{m-1}} & \xi^{(1)}_{\ell_1} & \xi^{(1)}_{\ell_2} & \ldots & \xi^{(1)}_{\ell_{m+1}} \\ \vdots & \vdots & \ldots & \vdots & \vdots & \vdots & \ldots & \vdots \\ \xi^{(m)}_{k_1} & \xi^{(m)}_{k_2} & \ldots & \xi^{(m)}_{k_{m-1}} & \xi^{(m)}_{\ell_1} & \xi^{(m)}_{\ell_2} & \ldots & \xi^{(m)}_{\ell_{m+1}} \\ \hdashline & & & & \xi^{(1)}_{\ell_1} & \xi^{(1)}_{\ell_2} & \ldots & \xi^{(1)}_{\ell_{m+1}} \\ & & 0 & & \vdots & \vdots & \ldots & \vdots \\ & & & & \xi^{(m)}_{\ell_1} & \xi^{(m)}_{\ell_2} & \ldots & \xi^{(m)}_{\ell_{m+1}} \end{array} \right| = 0 \ , \tag{6.17}$$

we obtain

$$\sum_{i=1}^{m+1} (-1)^\delta \xi_{Y_1} \xi_{Y_2} = 0 \ , \tag{6.18}$$

$$\delta = m + i - j \ , \tag{6.19}$$

where Y_1 is the Young diagram corresponding to $(k_1, \ldots, k_j, \ell_i, k_{j+1}, \ldots, k_{m-1})$ and Y_2 to $(\ell_1, \ldots, \ell_{i-1}, \ell_{i+1}, \ldots, \ell_{m+1})$, and where $k_j < \ell_i < k_{j+1}$ is satisfied. For all combinations of k_ν and ℓ_ν, Eq. (6.18) gives an infinite number of constraints on the ξ_Y's. These constraints are the Plücker relations.

On the other hand, if a function $f(t)$ is given as $f(t) = \sum_Y S_Y \xi_Y$, and if the ξ_Y's satisfy all the Plücker relations, then $f(t)$ becomes the τ-function, that is, $f(t)$ is written in the form of Eq. (6.2). For example, if $\xi_\phi = 1$, the matrix Ξ in the τ-function may be expressed as

$$
\Xi = \left(\begin{array}{cccc} \multicolumn{4}{c}{\overbrace{\hphantom{1 \quad 1 \quad \ddots \quad 1}}^{m}} \\ 1 & & & \\ & 1 & & 0 \\ & & \ddots & \\ 0 & & & 1 \\ \cdots & \xi_{\tiny\yng(1,1,1)} & -\xi_{\tiny\yng(1,1)} & \xi_{\tiny\yng(1)} \\ \cdots & \xi_{\tiny\yng(2,1,1)} & -\xi_{\tiny\yng(2,1)} & \xi_{\tiny\yng(2)} \\ \cdots & \vdots & \vdots & \vdots \end{array} \right) . \tag{6.20}
$$

We have shown that the ξ_Y's in Eq. (6.8) satisfy an infinite number of bilinear relations. We now show that the τ-function itself also satisfies partial differential equations with the same form as the Plücker relations.

From the definition of $\tau(t)$, Eq. (6.2), we may write

$$
\tau(t+s) = \det(\Xi_0{}^t \mathrm{e}^{\eta(t,\Lambda)}\, \mathrm{e}^{\eta(s,\Lambda)}\, \Xi) \ . \tag{6.21}
$$

If we denote

$$
\Xi(s) = \mathrm{e}^{\eta(s,\Lambda)}\, \Xi \ , \tag{6.22}
$$

Eq. (6.21) is rewritten as

$$
\tau(t+s) = \sum_Y S_Y(t)\, \xi_Y(s) \ , \tag{6.23}
$$

where the $\xi_Y(s)$'s are in the same form as Eq. (6.7) and satisfy all the Plücker relations with parameters $s = (s_1,\, s_2,\, s_3, \ldots)$. Applying $S_Y(\tilde{\partial}_t)$ to Eq. (6.23) and using the orthogonality condition of S_Y (see Appendix),

$$
S_Y(\tilde{\partial}_t) S_{Y'}(t)|_{t=0} = \delta_{YY'} \ , \tag{6.24}
$$

we obtain

$$
\begin{aligned}
\xi_Y(s) &= S_Y(\tilde{\partial}_t)\tau(t+s)|_{t=0} \\
&= S_Y(\tilde{\partial}_s)\tau(t+s)|_{t=0} \\
&= S_Y(\tilde{\partial}_s)\tau(s) \ .
\end{aligned} \tag{6.25}
$$

Substitution of Eq. (6.25) into Eq. (6.18) yields

$$
\sum (-1)^\delta \{S_{Y_1}(\tilde{\partial}_t)\tau(t)\}\{S_{Y_2}(\tilde{\partial}_t)\tau(t)\} = 0 \ , \tag{6.26}
$$

where δ is given by Eq. (6.19). Equation (6.26) is the bilinear equation for the τ-function. For example, Eq. (6.15) gives

$$S_\phi(\tilde{\partial}_t)\tau(t)\, S_{\begin{smallmatrix}\square&\square\\ \square&\square\end{smallmatrix}}(\tilde{\partial}_t)\tau(t) - S_{\square}(\tilde{\partial}_t)\tau(t)\, S_{\begin{smallmatrix}\square&\square\\ \square&\end{smallmatrix}}(\tilde{\partial}_t)\tau(t) + S_{\square\square}(\tilde{\partial}_t)\tau(t)\, S_{\begin{smallmatrix}\square\\ \square\end{smallmatrix}}(\tilde{\partial}_t)\tau(t) = 0\ . \tag{6.27}$$

Noticing that $S_\phi = 1$, $S_{\square} = t_1$, $S_{\square\square} = t_1^2/2 + t_2$, $S_{\begin{smallmatrix}\square\\ \square\end{smallmatrix}} = t_1^2/2 - t_2$, $S_{\begin{smallmatrix}\square&\square\\ \square&\end{smallmatrix}} = t_1^3/3 - t_3$ and $S_{\begin{smallmatrix}\square&\square\\ \square&\square\end{smallmatrix}} = t_1^4/12 - t_1 t_3 + t_2^2$, we obtain

$$\begin{aligned} \tau(t)\ \frac{1}{12}(\frac{\partial^4}{\partial t_1^4} - 4\frac{\partial^2}{\partial t_1 \partial t_3} + 3\frac{\partial^2}{\partial t_2^2})\tau(t)\ &-\ \frac{\partial}{\partial t_1}\tau(t)\ \frac{1}{3}(\frac{\partial^3}{\partial t_1^3} - \frac{\partial}{\partial t_3})\tau(t) \\ &+\ \frac{1}{2}(\frac{\partial^2}{\partial t_1^2} + \frac{\partial}{\partial t_2})\tau(t)\ \frac{1}{2}(\frac{\partial^2}{\partial t_1^2} - \frac{\partial}{\partial t_2})\tau(t) = 0\ , \end{aligned} \tag{6.28}$$

which is essentially the same as the bilinear form of the KP equation.

It is also possible to directly derive the bilinear equations which are written by Hirota's operators defined by Eq. (1.11). For the purpose, we evaluate

$$\sum_{\ell_\nu} a_1^{\ell_1} a_2^{\ell_2} \ldots a_m^{\ell_m} S_Y(t) = \sum_{\ell_\nu} a_1^{\ell_1} a_2^{\ell_2} \ldots a_m^{\ell_m} \begin{vmatrix} p_{\ell_1} & p_{\ell_2} & \cdots & p_{\ell_m} \\ p_{\ell_1 - 1} & p_{\ell_2 - 1} & \cdots & p_{\ell_m - 1} \\ \vdots & \vdots & \cdots & \vdots \\ p_{\ell_1 - m+1} & p_{\ell_2 - m+1} & \cdots & p_{\ell_m - m+1} \end{vmatrix}\ ,$$

where a_1, a_2, $\ldots$, a_m are arbitrary parameters and the summation is taken over all possible combinations of m nonnegative numbers. In the following, we assume that the set of numbers $(\ell_1,\ \ell_2,\ \ldots,\ \ell_m)$ corresponding to the Young diagram Y need not satisfy the condition $\ell_1 < \ell_2 < \ldots < \ell_m$. Even though the assumption may change the signs of S_Y and ξ_Y defined by Eqs. (6.6) and (6.7), respectively, the essence of the argument below is still kept.

We have

$$\begin{aligned} &\sum_{\ell_\nu} a_1^{\ell_1} a_2^{\ell_2} \ldots a_m^{\ell_m} S_Y(t) \\ &= \sum_{\ell_\nu} \begin{vmatrix} a_1^{\ell_1} p_{\ell_1} & a_2^{\ell_2} p_{\ell_2} & \cdots & a_m^{\ell_m} p_{\ell_m} \\ a_1 a_1^{\ell_1 - 1} p_{\ell_1 - 1} & a_2 a_2^{\ell_2 - 1} p_{\ell_2 - 1} & \cdots & a_m a_m^{\ell_m - 1} p_{\ell_m - 1} \\ \vdots & \vdots & \cdots & \vdots \\ a_1^{m-1} a_1^{\ell_1 - m+1} p_{\ell_1 - m+1} & a_2^{m-1} a_2^{\ell_2 - m+1} p_{\ell_2 - m+1} & \cdots & a_m^{m-1} a_m^{\ell_m - m+1} p_{\ell_m - m+1} \end{vmatrix} \end{aligned}$$

$$
= \begin{vmatrix}
\sum_{\ell_1=0}^{\infty} a_1^{\ell_1} p_{\ell_1} & \sum_{\ell_2=0}^{\infty} a_2^{\ell_2} p_{\ell_2} & \dots & \sum_{\ell_m=0}^{\infty} a_m^{\ell_m} p_{\ell_m} \\
a_1 \sum_{\ell_1=0}^{\infty} a_1^{\ell_1-1} p_{\ell_1-1} & a_2 \sum_{\ell_2=0}^{\infty} a_2^{\ell_2-1} p_{\ell_2-1} & \dots & a_m \sum_{\ell_m=0}^{\infty} a_m^{\ell_m-1} p_{\ell_m-1} \\
\vdots & \vdots & \dots & \vdots \\
a_1^{m-1} \sum_{\ell_1=0}^{\infty} a_1^{\ell_1-m+1} p_{\ell_1-m+1} & a_2^{m-1} \sum_{\ell_2=0}^{\infty} a_2^{\ell_2-m+1} p_{\ell_2-m+1} & \dots & a_m^{m-1} \sum_{\ell_m=0}^{\infty} a_m^{\ell_m-m+1} p_{\ell_m-m+1}
\end{vmatrix} .
$$

Noticing Eq. (3.3) and that $p_n = 0$ for $n < 0$, we find

$$
\begin{aligned}
\sum_{\ell_\nu} a_1^{\ell_1} a_2^{\ell_2} \dots a_m^{\ell_m} S_Y(t) &= \begin{vmatrix}
\mathrm{e}^{\eta(t,a_1)} & \mathrm{e}^{\eta(t,a_2)} & \dots & \mathrm{e}^{\eta(t,a_m)} \\
a_1 \mathrm{e}^{\eta(t,a_1)} & a_2 \mathrm{e}^{\eta(t,a_2)} & \dots & a_m \mathrm{e}^{\eta(t,a_m)} \\
\vdots & \vdots & \dots & \vdots \\
a_1^{m-1} \mathrm{e}^{\eta(t,a_1)} & a_2^{m-1} \mathrm{e}^{\eta(t,a_2)} & \dots & a_m^{m-1} \mathrm{e}^{\eta(t,a_m)}
\end{vmatrix} \\
&= \Delta(a_1, a_2, \dots, a_m) \exp \sum_{j=1}^{m} \eta(t, a_j) \ , \qquad (6.29)
\end{aligned}
$$

where

$$
\Delta(a_1, a_2, \dots, a_m) \equiv \prod_{1 \le i < j \le m} (a_j - a_i) \qquad (6.30)
$$

is the Vandermonde's determinant.

Let us introduce a function $\zeta(t; a_1, a_2, \dots, a_m)$ by

$$
\zeta(t; a_1, a_2, \dots, a_m) = \Delta(a_1, a_2, \dots, a_m) \left(\exp \sum_{j=1}^{m} \eta(\tilde{\partial}_t, a_j) \right) \tau(t) \ . \qquad (6.31)
$$

From Eq. (6.29), we have

$$
\zeta(t; a_1, a_2, \dots, a_m) = \sum_{\ell_\nu} a_1^{\ell_1} a_2^{\ell_2} \dots a_m^{\ell_m} S_Y(\tilde{\partial}_t) \tau(t) \ . \qquad (6.32)
$$

By using Eq. (6.25), Eq. (6.32) is reduced to

$$
\zeta(t; a_1, a_2, \dots, a_m) = \sum_{\ell_\nu} a_1^{\ell_1} a_2^{\ell_2} \dots a_m^{\ell_m} \xi_Y(t) \ , \qquad (6.33)
$$

which shows that ζ plays the role of the generating function for $\xi_Y(t)$.

From Eqs. (6.18) and (6.33), we have the following identity:

$$
\begin{aligned}
&\sum_{i=1}^{m+1}(-1)^{i-1}\zeta(t;b_1,\dots,b_{m-1},a_i)\zeta(t;a_1,\dots,a_{i-1},a_{i+1},\dots,a_{m+1})\\
=&\sum_{i=1}^{m+1}(-1)^{i-1}\sum b_1^{k_1}\dots b_{m-1}^{k_{m-1}}a_i^{\ell_i}(-1)^{m-j-1}\xi_{Y_1}(t)\\
&\qquad\times\sum a_1^{\ell_1}\dots a_{i-1}^{\ell_{i-1}}a_{i+1}^{\ell_{i+1}}\dots a_{m+1}^{\ell_{m+1}}\xi_{Y_2}(t)\\
=&\sum b_1^{k_1}\dots b_{m-1}^{k_{m-1}}a_1^{\ell_1}\dots a_{m+1}^{\ell_{m+1}}\sum_{i=1}^{m+1}(-1)^{\delta}\xi_{Y_1}(t)\xi_{Y_2}(t)\\
=&0\ ,
\end{aligned}
\tag{6.34}
$$

where δ is defined by Eq. (6.19). It is noted that the Young diagrams Y_1 and Y_2 are the same as those in Eq. (6.18).

On the other hand, Eq. (6.31) gives

$$
\begin{aligned}
&\sum_{i=1}^{m+1}(-1)^{i-1}\zeta(t;b_1,\dots,b_{m-1},a_i)\zeta(t;a_1,\dots,a_{i-1},a_{i+1},\dots,a_{m+1})\\
=&\sum_{i=1}^{m+1}(-1)^{i-1}\Delta(b_1,\dots,b_{m-1},a_i)\exp(\sum_{n=1}^{m-1}\eta(\tilde{\partial}_t,b_n)+\eta(\tilde{\partial}_t,a_i))\tau(t)\\
&\qquad\times\Delta(a_1,\dots,a_{i-1},a_{i+1},\dots,a_{m+1})\exp(\sum_{n=1}^{m+1}\eta(\tilde{\partial}_t,a_n)-\eta(\tilde{\partial}_t,a_i))\tau(t)\ .
\end{aligned}
\tag{6.35}
$$

Hence we have

$$
\begin{aligned}
&\sum_{i=1}^{m+1}(-1)^{i-1}\Delta(b_1,\dots,b_{m-1},a_i)\Delta(a_1,\dots,a_{i-1},a_{i+1},\dots,a_{m+1})\\
&\times\exp(\frac{1}{2}\sum_{n=1}^{m-1}\eta(\tilde{\partial}_t,b_n)-\frac{1}{2}\sum_{n=1}^{m+1}\eta(\tilde{\partial}_t,a_n)+\eta(\tilde{\partial}_t,a_i))\exp(\frac{1}{2}\sum_{n=1}^{m-1}\eta(\tilde{\partial}_t,b_n)+\frac{1}{2}\sum_{n=1}^{m+1}\eta(\tilde{\partial}_t,a_n))\tau(t)\\
&\times\exp(-\frac{1}{2}\sum_{n=1}^{m-1}\eta(\tilde{\partial}_t,b_n)+\frac{1}{2}\sum_{n=1}^{m+1}\eta(\tilde{\partial}_t,a_n)-\eta(\tilde{\partial}_t,a_i))\exp(\frac{1}{2}\sum_{n=1}^{m-1}\eta(\tilde{\partial}_t,b_n)+\frac{1}{2}\sum_{n=1}^{m+1}\eta(\tilde{\partial}_t,a_n))\tau(t)\\
&=0\ .
\end{aligned}
\tag{6.36}
$$

Noticing that

$$
\begin{aligned}
&\exp(\frac{1}{2}\sum_{n=1}^{m-1}\eta(\tilde{\partial}_t,b_n)+\frac{1}{2}\sum_{n=1}^{m+1}\eta(\tilde{\partial}_t,a_n))\tau(t)\\
=&\tau(t_1+\frac{1}{2}(\sum_{n=1}^{m-1}b_n+\sum_{n=1}^{m+1}a_n),\dots,t_j+\frac{1}{2j}(\sum_{n=1}^{m-1}b_n^j+\sum_{n=1}^{m+1}a_n^j),\dots)\ ,
\end{aligned}
\tag{6.37}
$$

and rewriting t_j for $t_j + \frac{1}{2j}(\sum_{n=1}^{m-1} b_n^j + \sum_{n=1}^{m+1} a_n^j)$, we have from Eq. (6.36) that

$$\begin{aligned}
&\sum_{i=1}^{m+1}(-1)^{i-1}\Delta(b_1,\ldots,b_{m-1},a_i)\Delta(a_1,\ldots,a_{i-1},a_{i+1},\ldots,a_{m+1})\\
&\times \exp(\sum_{n=1}^{m-1}\eta(\frac{1}{2}\tilde{\partial}_s,b_n)+\eta(\frac{1}{2}\tilde{\partial}_s,a_i))\exp(\sum_{n=1}^{m+1}\eta(-\frac{1}{2}\tilde{\partial}_s,a_n)+\eta(\frac{1}{2}\tilde{\partial}_s,a_i))\tau(t+s)\tau(t-s)\Big|_{s=0}\\
=&0\ .
\end{aligned} \tag{6.38}$$

Then, by means of Eq. (6.29), we obtain

$$\begin{aligned}
\sum_{i=1}^{m+1}(-1)^{i-1}\sum b_1^{k_1}\ldots b_{m-1}^{k_{m-1}}a_1^{\ell_1}\ldots a_{m+1}^{\ell_{m+1}}S_{Y_1}(\frac{1}{2}\tilde{\partial}_s)S_{Y_2}(-\frac{1}{2}\tilde{\partial}_s)\\
\times\tau(t+s)\tau(t-s)|_{s=0}=0\ .
\end{aligned} \tag{6.39}$$

Since each of the coefficient of $b_1^{k_1}\ldots b_{m-1}^{k_{m-1}}a_1^{\ell_1}\ldots a_{m+1}^{\ell_{m+1}}$ has to be zero, we get

$$\sum_{i=1}^{m+1}(-1)^{i-1}S_{Y_1}(\frac{1}{2}\tilde{\partial}_s)S_{Y_2}(-\frac{1}{2}\tilde{\partial}_s)\tau(t+s)\tau(t-s)\Big|_{s=0}=0\ . \tag{6.40}$$

If Hirota's operators defined by Eq. (1.11) are employed, Eq. (6.40) reduces to

$$\sum_{i=1}^{m+1}(-1)^{i-1}S_{Y_1}(\frac{1}{2}\tilde{D}_t)S_{Y_2}(-\frac{1}{2}\tilde{D}_t)\tau(t)\cdot\tau(t)=0\ . \tag{6.41}$$

For example, the Plücker relation (6.15) gives

$$(4D_{t_1}D_{t_3}-{D_{t_1}}^4-3{D_{t_2}}^2)\tau(t)\cdot\tau(t)=0\ , \tag{6.42}$$

which is nothing but the bilinear form of the KP equation, Eq. (1.9). Similarly, Eqs. (6.16a)$\sim$(6.16d) give

$$({D_{t_1}}^3D_{t_2}-3D_{t_1}D_{t_4}+2D_{t_2}D_{t_3})\tau(t)\cdot\tau(t)=0\ , \tag{6.43a}$$

$$({D_{t_1}}^6+4{D_{t_1}}^3D_{t_3}-9{D_{t_1}}^2{D_{t_2}}^2+36D_{t_2}D_{t_4}-32{D_{t_3}}^2)\tau(t)\cdot\tau(t)=0\ , \tag{6.43b}$$

$$D_{t_1}({D_{t_1}}^2D_{t_4}-2D_{t_1}D_{t_2}D_{t_3}+{D_{t_2}}^3)\tau(t)\cdot\tau(t)=0\ , \tag{6.43c}$$

$$\begin{aligned}
&({D_{t_1}}^8-20{D_{t_1}}^5D_{t_3}+15{D_{t_1}}^4{D_{t_2}}^2-72{D_{t_1}}^2D_{t_2}D_{t_4}+64{D_{t_1}}^2{D_{t_3}}^2\\
&\qquad+48D_{t_1}{D_{t_2}}^2D_{t_3}-36{D_{t_2}}^4)\tau(t)\cdot\tau(t)=0\ ,
\end{aligned} \tag{6.43d}$$

respectively. These equations form a subset of the KP hierarchy.

§7. Explicit expression of solutions by the τ-function

As mentioned in §3, the solutions of the Sato equation can be written in terms of certain derivatives of the τ-function. Then, through the transformations (4.3), the solutions of the generalized Lax equation are expressed by the τ-function. Moreover, from Eq. (5.16), the eigenfunction of the linear system (5.1)$\sim$(5.3) is also expressed by the τ-function. In this section, we give explicit expression of those functions.

Following Freeman and Nimmo[8)], we introduce the notation,

$$|\,\ell_1, \ell_2, \ldots, \ell_m| = \begin{vmatrix} h^{(1)}_{\ell_1} & h^{(2)}_{\ell_1} & \ldots & h^{(m)}_{\ell_1} \\ h^{(1)}_{\ell_2} & h^{(2)}_{\ell_2} & \ldots & h^{(m)}_{\ell_2} \\ \vdots & \vdots & \ldots & \vdots \\ h^{(1)}_{\ell_m} & h^{(2)}_{\ell_m} & \ldots & h^{(m)}_{\ell_m} \end{vmatrix} , \tag{7.1}$$

where $h_n^{(j)}$ satisfies Eq. (3.10). Then, the τ-function (6.1) and the solutions (3.14) of the Sato equation are simply written by

$$\tau(x;t) = |\,0, 1, \ldots, m-1| \ , \tag{7.2}$$

and

$$w_j(x;t) = (-1)^j \frac{|\,0, 1, \ldots, m-j-1, m-j+1, \ldots, m|}{|\,0, 1, \ldots, m-1|} , \tag{7.3}$$

respectively.

By using the Vandermonde's determinant defined by Eq. (6.30), the τ-function may also be expressed as

$$\tau(x;t) = \Delta(\partial_{x_1}, \partial_{x_2}, \ldots, \partial_{x_m})\ h_0^{(1)}(x_1;t) h_0^{(2)}(x_2;t) \ldots h_0^{(m)}(x_m;t)\Big|_{x_1,\ldots,x_m=x} . \tag{7.4}$$

If we differentiate Eq. (7.4) by t_ℓ, we obtain

$$\frac{\partial\tau}{\partial t_\ell} = \sum_{k=1}^{m} \partial^\ell_{x_k} \Delta(\partial_{x_1}, \ldots, \partial_{x_m}) h_0^{(1)}(x_1;t) \ldots h_0^{(m)}(x_m;t)\Bigg|_{x_1,\ldots,x_m=x} , \tag{7.5}$$

where we have used Eq. (3.10).

Let us operate $S_Y(\tilde{\partial}_t)$ on Eq. (7.4) for a certain Young diagram Y. From Eq. (A.20), we have

$$S_Y(\tilde{\partial}_t)\tau = \sum_\rho \chi_\rho^Y \frac{\partial_{t_1}^{\alpha_1}\ldots\partial_{t_m}^{\alpha_m}}{1^{\alpha_1}\ldots m^{\alpha_m}\alpha_1!\cdots\alpha_m!}\tau \ . \tag{7.6}$$

By using Eqs. (7.5) and (A.5), Eq. (7.6) is reduced to

$$\begin{aligned} S_Y(\tilde{\partial}_t)\tau = \frac{1}{m!}\sum_\rho \chi_\rho^Y h_\rho (\sum_{k=1}^m \partial_{x_k})^{\alpha_1}\ldots(\sum_{k=1}^m \partial_{x_k}^m)^{\alpha_m} \\ \times\ \Delta(\partial_{x_1},\ldots,\partial_{x_m})h_0^{(1)}(x_1;t)\ldots h_0^{(m)}(x_m;t)\Big|_{x_1,\ldots,x_m=x} \ . \end{aligned}\tag{7.7}$$

If we take $\epsilon_i = \partial_{x_i}$ in Eq. (A.15), we have

$$(\sum_{k=1}^m \partial_{x_k})^{\alpha_1}\ldots(\sum_{k=1}^m \partial_{x_k}^m)^{\alpha_m} = \sum_{|Y|=m} \chi_\rho^Y \tilde{S}_Y(\partial_x) \ , \tag{7.8}$$

where ∂_x means $(\partial_{x_1},\, \partial_{x_2},\, \ldots,\, \partial_{x_m})$. We then obtain

$$\begin{aligned} S_Y(\tilde{\partial}_t)\tau =& \frac{1}{m!}\sum_{|Y'|=m}\sum_\rho h_\rho \chi_\rho^Y \chi_\rho^{Y'} \\ &\times\ \tilde{S}_{Y'}(\partial_x)\Delta(\partial_{x_1},\ldots,\partial_{x_m})\ h_0^{(1)}(x_1;t)\ldots h_0^{(m)}(x_m;t)\Big|_{x_1,\ldots,x_m=x} \ . \end{aligned}\tag{7.9}$$

The orthogonality condition for the irreducible characters, Eq. (A.4), yields

$$S_Y(\tilde{\partial}_t)\tau = \tilde{S}_Y(\partial_x)\Delta(\partial_{x_1},\ldots,\partial_{x_m})h_0^{(1)}(x_1;t)\ldots h_0^{(m)}(x_m;t)\Big|_{x_1,\ldots,x_m=x} \ . \tag{7.10}$$

By taking $\epsilon_i = \partial_{x_i}$ in Eq. (A.13) and substituting $\tilde{S}_Y(\partial_x)$ into Eq. (7.10), we have

$$\begin{aligned} S_Y(\tilde{\partial}_t)\tau =& \begin{vmatrix} \partial_{x_1}^{\nu_m} & \partial_{x_1}^{\nu_{m-1}+1} & \ldots & \partial_{x_1}^{\nu_1+m-1} \\ \partial_{x_2}^{\nu_m} & \partial_{x_2}^{\nu_{m-1}+1} & \ldots & \partial_{x_2}^{\nu_1+m-1} \\ \vdots & \vdots & \ldots & \vdots \\ \partial_{x_m}^{\nu_m} & \partial_{x_m}^{\nu_{m-1}+1} & \ldots & \partial_{x_m}^{\nu_1+m-1} \end{vmatrix} h_0^{(1)}(x_1;t)\ldots h_0^{(m)}(x_m;t)\Big|_{x_1,\ldots,x_m=x} \\ =& \,|\,\nu_m, \nu_{m-1}+1, \ldots, \nu_1+m-1| \ , \end{aligned}\tag{7.11}$$

where $[\nu_1,\, \nu_2,\, \ldots,\, \nu_m]$ is the partition corresponding to the Young diagram Y.

If we consider the partition $[\underbrace{1,1,\ldots,1}_{j},\underbrace{0,0,\ldots,0}_{m-j}]$ or the Young diagram $\left.\begin{array}{|c|}\hline \ \\ \hline \ \\ \hline \ \\ \hline\end{array}\right\}j$, then Eq. (7.11) gives

$$S_{\left.\begin{array}{|c|}\hline \ \\ \hline \ \\ \hline \ \\ \hline\end{array}\right\}j}(\tilde{\partial}_t)\,\tau = |\,0, 1, \ldots, m-j-1, m-j+1, \ldots, m| \ . \tag{7.12}$$

Substituting Eq. (7.12) into Eq. (7.3), we find

$$w_j = (-1)^j \frac{1}{\tau} S_{\left.\begin{smallmatrix}\square\\ \square\\ \square\end{smallmatrix}\right\}j}(\tilde{\partial}_t)\, \tau \ . \tag{7.13}$$

This is an explicit expression for the solutions of the Sato equation in terms of the τ-function.

The expression (7.13) may be simplified by using some properties of the Schur function. From Eq. (A.18), we have

$$S_{\left.\begin{smallmatrix}\square\\ \square\\ \square\end{smallmatrix}\right\}j}(\tilde{\partial}_t) = \frac{1}{j!} \sum_{\rho} h_\rho \chi_\rho^{\left.\begin{smallmatrix}\square\\ \square\\ \square\end{smallmatrix}\right\}j} \Big(\frac{\partial}{\partial t_1}\Big)^{\alpha_1} \Big(\frac{\partial}{\partial t_2}\Big)^{\alpha_2} \cdots \Big(\frac{\partial}{\partial t_j}\Big)^{\alpha_j} \ , \tag{7.14}$$

and

$$\begin{aligned} S_{\underbrace{\square\square\square}_{j}}(-\tilde{\partial}_t) = \frac{1}{j!} \sum_{\rho} h_\rho \chi_\rho^{\overbrace{\square\square\square\square}^{j}} & \\ \times (-1)^{\alpha_1+\alpha_2+\cdots+\alpha_j} \Big(\frac{\partial}{\partial t_1}\Big)^{\alpha_1} \Big(\frac{\partial}{\partial t_2}\Big)^{\alpha_2} & \cdots \Big(\frac{\partial}{\partial t_j}\Big)^{\alpha_j} \ . \end{aligned} \tag{7.15}$$

By using the relation between the irreducible characters $\chi_\rho^{\square\square\square\square}$ and $\chi_\rho^{\begin{smallmatrix}\square\\ \square\\ \square\\ \square\end{smallmatrix}}$, Eq. (A.2), and noticing that

$$S_{\underbrace{\square\square\square}_{j}}(t) = p_j(t) \ , \tag{7.16}$$

we finally obtain

$$w_j = \frac{1}{\tau}\, p_j(-\tilde{\partial}_t)\, \tau \ , \tag{7.17}$$

which is the simplified expression for the solutions of the Sato equation. For example, we have

$$w_1 = -\frac{1}{\tau}\frac{\partial \tau}{\partial x} \ , \tag{7.18a}$$

$$w_2 = \frac{1}{2\tau}\Big(\frac{\partial^2 \tau}{\partial x^2} - \frac{\partial \tau}{\partial t_2}\Big) \ , \tag{7.18b}$$

$$w_3 = -\frac{1}{6\tau}\Big(\frac{\partial^3 \tau}{\partial x^3} - 3\frac{\partial^2 \tau}{\partial x \partial t_2} + 2\frac{\partial \tau}{\partial t_3}\Big) \ . \tag{7.18c}$$

The solutions of the generalized Lax equation can also be expressed in terms of the τ-function through the transformation (4.3). For example,

$$u_2 = \frac{\partial^2}{\partial x^2} \log \tau \ , \tag{7.19a}$$

$$u_3 = \frac{1}{2}(-\frac{\partial^3}{\partial x^3} + \frac{\partial^2}{\partial x \partial t_2}) \log \tau \ , \tag{7.19b}$$

$$u_4 = \frac{1}{6}(\frac{\partial^4}{\partial x^4} - 3\frac{\partial^3}{\partial x^2 \partial t_2} + 2\frac{\partial^2}{\partial x \partial t_3}) \log \tau - (\frac{\partial^2}{\partial x^2} \log \tau)^2 \ . \tag{7.19c}$$

Especially, Eq. (7.19a) is nothing but the dependent variable transformation (1.10) to reduce the KP equation to its bilinear form.

It is also possible to express the eigenfunction (5.16) in terms of the τ-function. Substitution of Eq. (7.17) into Eq. (5.16) yields

$$\begin{aligned}
\psi &= \frac{1}{\tau}\left\{\tau + \frac{p_1(-\tilde{\partial}_t)\tau}{\lambda} + \frac{p_2(-\tilde{\partial}_t)\tau}{\lambda^2} + \cdots\right\} e^{t_0+\lambda t_1+\lambda^2 t_2+\cdots} \\
&= \frac{1}{\tau}\left\{\left(\sum_{\nu_1=0}^{\infty} \frac{1}{\nu_1!}(\frac{-\partial_{t_1}}{\lambda})^{\nu_1} \sum_{\nu_2=0}^{\infty} \frac{1}{\nu_2!}(\frac{-\partial_{t_2}}{2\lambda^2})^{\nu_2} \cdots\right)\tau\right\} e^{t_0+\lambda t_1+\lambda^2 t_2+\cdots} \\
&= \frac{1}{\tau}\left\{\exp(-\frac{1}{\lambda}\partial_{t_1} - \frac{1}{2\lambda^2}\partial_{t_2} - \cdots)\tau\right\} e^{t_0+\lambda t_1+\lambda^2 t_2+\cdots} \ .
\end{aligned} \tag{7.20}$$

Thus we obtain

$$\psi = \frac{\tau(t_1 - \frac{1}{\lambda}, t_2 - \frac{1}{2\lambda^2}, \ldots)}{\tau(t_1, t_2, \ldots)} e^{t_0+\lambda t_1+\lambda^2 t_2+\cdots} \ . \tag{7.21}$$

§8. Concluding remarks

We have made an introduction to Sato theory in the preceding sections. It has been shown that the Sato equation generates the generalized Lax equation, the Zakharov-Shabat equation and the IST scheme. It has also been shown that an infinite number of nonlinear evolution equations (the KP hierarchy), of which the KP equation is the simplest nontrivial one, share solutions. The τ-function is the key function to express the solutions. By employing the results of the representation theory of groups, we have shown that the partial differential equations governing the τ-function are closely related to the Plücker relations and may be written in bilinear form. The solutions of the Sato equation, and consequently those of the generalized Lax equation and the eigenfunction in the IST scheme, are explicitly expressed by the τ-function.

Since the aim of this paper is the elementary introduction to Sato theory, we did not include any further results. The extension to the multi-component systems includes the nonlinear evolution equations such as the nonlinear Schrödinger equation and the sine-Gordon equation. The theory also applies to discrete systems such as the Toda equation. Date, Jimbo, Kashiwara and Miwa have extended this theory by using the method of field theory. In this approach, the relationship between the soliton equations and the infinite dimensional Lie algebra becomes apparent. Moreover, the vertex operator is shown to have a close relation to the Bäcklund transformation. Interested readers may refer to Refs. 10)$\sim$14) and 16).

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Appendix : Results from the representation theory of groups

In this appendix, we present some results of the representation theory of groups which are useful to discuss the structures of the τ-function.

A-1 *Irreducible character*

We consider the symmetric group $\mathcal{S}_m$, *i.e.*, the permutation group of m numbers. The number of elements is $m!$.

Example The elements of $\mathcal{S}_3$ are

$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} , \quad \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} , \quad \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix} ,$$

$$\begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix} , \quad \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} , \quad \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix} .$$

The elements of $\mathcal{S}_m$ are classified into the classes $(1^{\alpha_1},\, 2^{\alpha_2},\, \ldots,\, m^{\alpha_m})$, where $\alpha_1 + 2\alpha_2 + \cdots + m\alpha_m = m$.

Example For $\mathcal{S}_3$, we have

$$\begin{array}{ll} (3^1) \quad \ni & \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} , \quad \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} , \\ (1^1, 2^1) \ni & \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix} , \quad \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix} , \quad \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} , \\ (1^3) \quad \ni & \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix} . \end{array}$$

Irreducible representation of $\mathcal{S}_m$ is characterized by the partition $[\lambda]$, where

$$[\lambda] = [\lambda_1, \lambda_2, \ldots, \lambda_m]$$

means the set of numbers satisfying

$$\lambda_1 + \lambda_2 + \cdots + \lambda_m = m \ ,$$

$$\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_m \geq 0 \ .$$

To each partition $[\lambda]$ there corresponds a Young diagram.

Example For $\mathcal{S}_3$, we have

$$
\begin{aligned}
3+0+0 &\longleftrightarrow [3,0,0] \text{ or } [3] \longleftrightarrow \ydiagram{3} \ , \\
2+1+0 &\longleftrightarrow [2,1,0] \text{ or } [2,1] \longleftrightarrow \ydiagram{2,1} \ , \\
1+1+1 &\longleftrightarrow [1,1,1] \text{ or } [1^3] \longleftrightarrow \ydiagram{1,1,1} \ .
\end{aligned}
$$

The partition has the one to one correspondence to the set of numbers (ℓ_1, ℓ_2, $\ldots$, ℓ_m) introduced in §6. Namely,

$$[\lambda] \longleftrightarrow (\ell_1, \ell_2 - 1, \ldots, \ell_m - m + 1) \longleftrightarrow (\ell_1, \ell_2, \ldots, \ell_m)$$

Example For $\mathcal{S}_3$, we have

$$
\begin{aligned}
[3] &\longleftrightarrow (0,0,3) \longleftrightarrow (0,1,5) \ , \\
[2,1] &\longleftrightarrow (0,1,2) \longleftrightarrow (0,2,4) \ , \\
[1^3] &\longleftrightarrow (1,1,1) \longleftrightarrow (1,2,3) \ .
\end{aligned}
$$

If a is an element of $\mathcal{S}_m$ belonging to the class (1^{α_1}, 2^{α_2}, $\ldots$, m^{α_m}), then we have a relation,

$$t_1^{\alpha_1} t_2^{\alpha_2} \ldots t_m^{\alpha_m} = \frac{1}{1^{\alpha_1} 2^{\alpha_2} \ldots m^{\alpha_m}} \sum_{|Y|=m} \chi^Y(a) S_Y(t) \ , \tag{A.1}$$

where S_Y is defined by Eq. (6.6), $\chi^Y(a)$ is the irreducible character, $|Y|$ is the size (the number of $\square$'s) of the Young diagram, and the summation is taken over all the Young diagrams of size m.

Example If $a \in (3^1)$, then we have

$$t_3{}^1 = \frac{1}{3^1} \left\{ \chi^{\ydiagram{3}}(a) S_{\ydiagram{3}}(t) + \chi^{\ydiagram{2,1}}(a) S_{\ydiagram{2,1}}(t) + \chi^{\ydiagram{1,1,1}}(a) S_{\ydiagram{1,1,1}}(t) \right\} \ .$$

The values of the irreducible characters are determined by Eq. (A.1). The values for all elements belonging to the same class are identical. Hence $\chi^Y(a)$ may be written as χ^Y_ρ, where ρ is the class to which a belongs.

Example For $\mathcal{S}_3$, we have the following chart of the irreducible characters:

class	number of elements	irreducible character		
		$\chi^{\ydiagram{3}}$	$\chi^{\ydiagram{2,1}}$	$\chi^{\ydiagram{1,1,1}}$
(1^3)	1	1	2	1
$(1^1, 2^1)$	3	1	0	-1
(3^1)	2	1	-1	1

From the definition of the irreducible characters, we especially have

$$\chi_\rho^{\overbrace{\square\square\square\square}^{m}} = (-1)^{m-\sum\limits_{j=1}^{m}\alpha_j}\chi_\rho^{\left.\begin{smallmatrix}\square\\\square\\\square\\\square\end{smallmatrix}\right\}m} = 1 \ . \tag{A.2}$$

There exists the orthogonality relation of the first kind among the irreducible characters,

$$\frac{1}{m!}\sum_{a\in\mathcal{S}_m}\chi^Y(a)\chi^{Y'}(a) = \delta_{YY'} \ , \tag{A.3}$$

where the summation is taken over all the elements of $\mathcal{S}_m$. Since the irreducible characters are identical for all elements belonging to the same class ρ, Eq. (A.3) may be rewritten by

$$\frac{1}{m!}\sum_\rho h_\rho \chi_\rho^Y \chi_\rho^{Y'} = \delta_{YY'} \ , \tag{A.4}$$

where

$$h_\rho = \frac{m!}{1^{\alpha_1}2^{\alpha_2}\cdots m^{\alpha_m}\alpha_1!\alpha_2!\cdots\alpha_m!} \tag{A.5}$$

is the number of elements belonging to the class ρ.

A-2 *Schur function*

Let us define $F(x)$ and φ_j, $j = 1, 2, \ldots,$ by

$$F(x) = \prod_{\nu=1}^{m}(1-\epsilon_\nu x)^{-1} = 1 + \varphi_1 x + \varphi_2 x^2 + \cdots \ . \tag{A.6}$$

By equating the same powers of x in Eq. (A.6), we obtain

$$\varphi_j(\epsilon) = \varphi_j(\epsilon_1, \epsilon_2, \ldots, \epsilon_m) = \sum_{\lambda_1+\lambda_2+\cdots+\lambda_m=j}\epsilon_1^{\lambda_1}\epsilon_2^{\lambda_2}\ldots\epsilon_m^{\lambda_m} \ . \tag{A.7}$$

On the other hand, we have

$$F'(x)/F(x) = t_1 + 2t_2 x + 3t_3 x^2 + \cdots \ , \tag{A.8}$$

where

$$t_\ell = \frac{1}{\ell}(\epsilon_1^\ell + \epsilon_2^\ell + \cdots + \epsilon_m^\ell) \ . \tag{A.9}$$

Integration of Eq. (A.8) yields

$$F(x) = \exp \sum_{\ell=1}^{\infty} t_\ell x^\ell \ , \tag{A.10}$$

which, by means of Eq. (3.3), reduces to

$$F(x) = \sum_{\ell=0}^{\infty} p_\ell x^\ell \ . \tag{A.11}$$

Then, from Eqs. (A.6) and (A.11), we find

$$\varphi_j(\epsilon) = p_j(t) \ . \tag{A.12}$$

The Schur function is usually defined by

$$\tilde{S}_{[\nu]}(\epsilon) = \frac{\begin{vmatrix} \epsilon_1^{\nu_m} & \epsilon_1^{\nu_{m-1}+1} & \cdots & \epsilon_1^{\nu_1+m-1} \\ \epsilon_2^{\nu_m} & \epsilon_2^{\nu_{m-1}+1} & \cdots & \epsilon_2^{\nu_1+m-1} \\ \vdots & \vdots & \cdots & \vdots \\ \epsilon_m^{\nu_m} & \epsilon_m^{\nu_{m-1}+1} & \cdots & \epsilon_m^{\nu_1+m-1} \end{vmatrix}}{\begin{vmatrix} \epsilon_1^0 & \epsilon_1^1 & \cdots & \epsilon_1^{m-1} \\ \epsilon_2^0 & \epsilon_2^1 & \cdots & \epsilon_2^{m-1} \\ \vdots & \vdots & \cdots & \vdots \\ \epsilon_m^0 & \epsilon_m^1 & \cdots & \epsilon_m^{m-1} \end{vmatrix}} \ , \tag{A.13}$$

where $[\nu]$ means the partition $[\nu_1, \nu_2, \ldots, \nu_m]$ (See for example Ref. 17)). It is noted that the denominator is the Vandermonde's determinant $\Delta(\epsilon_1, \epsilon_2, \ldots, \epsilon_m)$. Equation (A.13) may be rewritten as

$$\tilde{S}_Y(\epsilon) = \begin{vmatrix} \varphi_{\nu_m} & \varphi_{\nu_{m-1}+1} & \cdots & \varphi_{\nu_1+m-1} \\ \varphi_{\nu_m-1} & \varphi_{\nu_{m-1}} & \cdots & \varphi_{\nu_1+m-2} \\ \vdots & \vdots & \cdots & \vdots \\ \varphi_{\nu_m-m+1} & \varphi_{\nu_{m-1}-m+2} & \cdots & \varphi_{\nu_1} \end{vmatrix} \ , \tag{A.14}$$

where Y is the Young diagram corresponding to the partition $[\nu]$.

By substituting Eq. (A.12) into Eq. (A.14), we find that $\tilde{S}_Y(\epsilon)$ is equal to $S_Y(t)$ defined by Eq. (6.6). Then, by using Eq. (A.9), we see that Eq. (A.1) is equivalent to

$$(\sum_{\ell=1}^{m} \epsilon_\ell)^{\alpha_1} (\sum_{\ell=1}^{m} \epsilon_\ell^2)^{\alpha_2} \cdots (\sum_{\ell=1}^{m} \epsilon_\ell^m)^{\alpha_m} = \sum_{|Y|=m} \chi_\rho^Y \tilde{S}_Y(\epsilon) \ . \tag{A.15}$$

A-3 *Expansion of functions in* $S_Y(t)$

An arbitrary analytic function $f(t)$ may be written by

$$f(t)=\sum_{m=0}^{\infty}\sum_{\alpha_1+2\alpha_2+\cdots+m\alpha_m=m}C^{(m)}(\alpha_1,\alpha_2,\ldots,\alpha_m)t_1^{\alpha_1}(2t_2)^{\alpha_2}(3t_3)^{\alpha_3}\ldots(mt_m)^{\alpha_m}\ , \tag{A.16}$$

where $C^{(m)}(\alpha_1,\alpha_2,\ldots,\alpha_m)$ is constant. It is clear that all the elements of the set $\{t_1^{\alpha_1}$ $(2t_2)^{\alpha_2}$ $(3t_3)^{\alpha_3}\ldots\}$ are linearly independent.

From Eq. (A.1), we have

$$t_1^{\alpha_1}(2t_2)^{\alpha_2}\ldots(mt_m)^{\alpha_m}=\sum_{|Y|=m}\chi_\rho^Y\,S_Y(t)\ . \tag{A.17}$$

The orthogonality relation (A.4) gives

$$S_Y(t)=\frac{1}{m!}\sum_\rho h_\rho\chi_\rho^Y\,t_1^{\alpha_1}(2t_2)^{\alpha_2}\ldots(mt_m)^{\alpha_m}\ . \tag{A.18}$$

Equations (A.17) and (A.18) show that there exists a linear transformation between $\{S_Y(t);|Y|=m\}$ and $\{t_1^{\alpha_1}\ (2t_2)^{\alpha_2}\ \ldots(mt_m)^{\alpha_m};\alpha_1+2\alpha_2+\cdots+m\alpha_m=m\}$. Hence we find that $f(t)$ can be uniquely expressed in $S_Y(t)$ as

$$f(t)=\sum_Y S_Y(t)c_Y\ . \tag{6.10}$$

Substituting Eq. (A.5) into Eq. (A.18), we have

$$S_Y(t)=\sum_\rho\chi_\rho^Y\frac{t_1^{\alpha_1}t_2^{\alpha_2}\cdots t_m^{\alpha_m}}{\alpha_1!\alpha_2!\cdots\alpha_m!}\ , \tag{A.19}$$

which gives

$$S_Y(\tilde{\partial}_t)=\sum_\rho\chi_\rho^Y\frac{\partial_{t_1}^{\alpha_1}\partial_{t_2}^{\alpha_2}\cdots\partial_{t_m}^{\alpha_m}}{1^{\alpha_1}2^{\alpha_2}\cdots m^{\alpha_m}\alpha_1!\alpha_2!\cdots\alpha_m!}\ . \tag{A.20}$$

Then, using the orthogonality relation (A.4), we find

$$S_Y(\tilde{\partial}_t)S_{Y'}(t)|_{t=0}=\delta_{YY'}\ . \tag{6.24}$$

Furthermore, if we apply Eq. (6.24) to Eq. (6.10), we obtain

$$c_Y=S_Y(\tilde{\partial}_t)\,f(t)|_{t=0}\ . \tag{6.11}$$

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Figure Captions

Fig. 1 : The chain of cells to construct the Maya diagram.

Fig. 2 : The Maya diagram expressing the set of numbers (2, 3, 5, 7).

Fig. 3 : The Maya diagram expressing the vacuum state.

Fig. 4 : The procedure to construct the Young diagram corresponding to the Maya diagram of Fig. 2 and the resultant Young diagram.

Fig. 1 :

⋯	−5	−4	−3	−2	−1	0	1	2	3	4	5	⋯
⋯												⋯

Fig. 2 :

⋯	−5	−4	−3	−2	−1	0	1	2	3	4	5	⋯
⋯	○	○			○	○		○		○		⋯

Fig. 3 :

⋯	−5	−4	−3	−2	−1	0	1	2	3	4	5	⋯
⋯	○	○	○	○	○	○						⋯

Fig. 4 :