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Asymptotic solutions to a fuzzy elementary cellular automaton of rule number 38

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Abstract

Fuzzy cellular automaton (CA) is a dynamical system with a continuous state value embedding a CA with a discrete state value. We investigate a fuzzy CA obtained from an elementary CA of rule number 38. Its asymptotic solutions are classified into two types. One is a solution where stable propagating waves exist, and the other is a static uniform solution of constant value.

Keywords fuzzy system, cellular automaton, asymptotic solution

Research Activity Group Applied Integrable Systems

1. Introduction

A cellular automaton (CA) is a dynamical system with a finite set of state values in discrete coordinates. As a consequence of their concise form and rich dynamics, the theory and applications of CA have received a great deal of interest [1]. One of the simplest configurations is the elementary cellular automaton (ECA) of which the binary state value at the next time is determined from three neighbors in one-dimensional space sites at the current time as follows:

$$u_j^{n+1} = f(u_{j-1}^n, u_j^n, u_{j+1}^n), \tag{1}$$

where j denotes an integer space site, n is an integer time step, and u is a binary state value ($u \in \{0,1\}$). Since f is binary valued with three binary arguments, it can be defined by the following rule table where $b_k \in \{0,1\}$.

$x\,y\,z$	111	110	101	100	011	010	001	000
$f(x,y,z)$	b_1	b_2	b_3	b_4	b_5	b_6	b_7	b_8

(2)

Thus, there are 256 different rules defined by the above table with each ECA being identified uniquely by the rule number $(b_1 b_2 \ldots b_8)_2$. ECA has been mathematically analyzed from various viewpoints, including investigations of the structure of the solutions, as well as studies of the statistical properties of solution patterns.

The systems obtained by embedding CA in continuous real or rational background is generally referred to as "fuzzy" CA [2]. For example, fuzzy ECA is defined in the form of (1) where j and n are integers and $u \in [0,1]$. There are infinite variations on f since its necessary condition is $[0,1]^3 \to [0,1]$ together with $\{0,1\}^3 \to \{0,1\}$. This condition means fuzzification together with embedding of the CA. One of the common forms of fuzzy CA is defined by using the polynomial [3,4]. For example, if we define

$$f(x,y,z) = xyz, \tag{3}$$

then (1) becomes a fuzzy ECA of rule number 128.

Since fuzzy CA is a continuous extension of CA, it has been used as an application models to express an intermediate state value among the original discrete values [5,6]. Moreover, there exists another important significance for fuzzy CA from the theoretical viewpoint. Since it embeds CA in the continuous range, continuous solutions to fuzzy CA propose a rich comprehension to discrete ones to its original CA [7,8]. We discuss asymptotic solutions to a fuzzy CA obtained from an ECA in this article. Its range is a continuous interval $[0,1]$, and it also proposes solutions to the ECA as a special case of a discrete range $\{0,1\}$.

Let us consider the following equation:

$$\begin{aligned} u_j^{n+1} &= f(u_{j-1}^n, u_j^n, u_{j+1}^n), \\ f(x,y,z) &= y + z - xy - 2yz + xyz \\ &= (1-x)y(1-z) + (1-y)z, \end{aligned} \tag{4}$$

where j denotes an integer space site and n an integer time step. Space is finite and the domain is $0 \le j < K$ with a periodic boundary condition $u_{j+K}^n = u_j^n$. We can easily show that the value of solutions to (4) can be closed in $u \in \{0,1\}$, $(0,1)$, or $[0,1]$. If $u \in \{0,1\}$, then (4) is equivalent to the following rule table.

$x\,y\,z$	111	110	101	100	011	010	001	000
$f(x,y,z)$	0	0	1	0	0	1	1	0

(5)

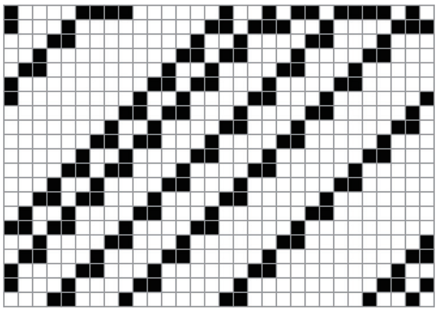

Fig. 1. Example of a solution to ECA38 for $K = 30$. The space coordinates j and n are rightward and downward, respectively. ECA: elementary cellular automaton.

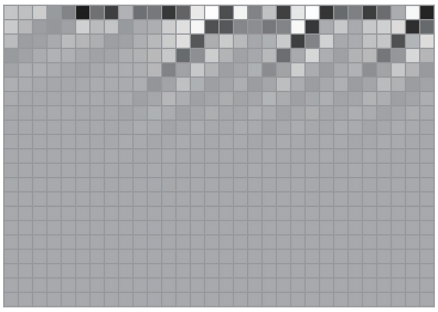

Fig. 2. Example of a solution to fuzzy ECA defined by (4). State values are shown in the grayscale from white (0) to black (1). ECA: elementary cellular automaton.

It is the evolution rule of ECA of rule number 38, and an example of time evolution is shown in Fig. 1.

We can consider (4) as a fuzzy CA obtained by extending as the state value of ECA38 to be continuous in the range $[0, 1]$. An example of the evolution from random initial data in $[0, 1]$ is shown in Fig. 2. This figure suggests that the random initial data converge to a uniform state for $n \to \infty$. If we denote a uniform solution as $u_j^n = v_n$, then v_n satisfies the following mapping:

$$v_{n+1} = f(v_n, v_n, v_n) = v_n(1 - v_n)(2 - v_n). \qquad (6)$$

This mapping can be closed in $[0, 1]$, and there is only one stable fixed point $\omega = (3 - \sqrt{5})/2$ satisfying $\omega = f(\omega, \omega, \omega)$, that is, $\omega^2 - 3\omega + 1 = 0$ and $0 \le \omega \le 1$. Numerical computations starting from random initial data as shown in Fig. 2 imply that the asymptotic solution converges to the uniform state as $u_j^n \to \omega$. However, if we restrict the initial data to be binary, that is, 0 or 1, then the uniform state $u \equiv \omega$ cannot appear considering the rule Table (5) and stable triangular waves with value 1 propagate in $-j$ direction as shown in Fig. 1.

The asymptotic behaviors of both solutions differ significantly. One is uniform and static and the other nonuniform and moving stably. In this article, we discuss and classify the asymptotic solutions to (4) closed in $[0, 1]$. The remainder of this article is organized as follows. In Section 2, asymptotic solutions including at least one 0 or 1 are discussed. We call this type of solution "type A." In Section 3, those solutions closed in $(0, 1)$ including neither 0 nor 1 are discussed. We call this type of solution "type B." In Section 4, we give concluding remarks.

Other researchers have also considered convergence of solutions to fuzzy CA. Betel and Flocchini proposed a general theorem about fuzzy CA of which the evolution rule is in a form of the weighted average of two sites [7]. They showed that various fuzzy CAs with 3 neighbors give a uniform static solution asymptotically using their theorem and a transformation of the map defined by the evolution rule. Fukuda and Watanabe classified some fuzzy CAs according to the types of asymptotic solutions using the Gröbner basis [9]. Mingarelli studied a fuzzy CA derived from ECA of rule number 110 and showed that a solution from initial data with non-zero value on one site in a zero background converges to a uniform solution [10].

Note that the evolution rule defined by (4) can not be treated by the theory proposed in [7]. Though the form is similar to that of a weighted average, the minimum and maximum values of sites are not monotonic and hence the analysis is more challenging.

2. Asymptotic solutions including 0 or 1

In this section, we discuss asymptotic solutions of type A, that is, those where there exists at least one site such that $u = 0$ or 1 for $n \to \infty$. Once $0 < u_j^n < 1$ holds for any j at a certain n, the solution always satisfies $0 < u < 1$ thereafter since (4) can be considered to be an interpolation between y and $1 - y$ with weights $(1 - x)(1 - z)$ and z. Therefore, there exists at least one site with value 0 or 1 at arbitrary $n \gg 0$ for asymptotic solutions of type A; otherwise, solutions are of type B. Note that a special case of the uniform solution $u \equiv 0$ is also of type A, but we exclude this trivial case from the discussion below.

Assume that the symbol $*$ denotes an arbitrary value x satisfying $0 < x < 1$. Then, the rule table for the different combinations of values other than (5) is given as follows:

11*	10*	01*	00*	1*1	1*0	0*1	0*0
0	*	*	*	*	0	*	*

*11	*10	*01	*00
0	*	1	0

1**	0**	*1*	*0*	**1	**0	***
*	*	*	*	*	*	*

(7)

Considering all local patterns ** with 0 or 1 attached to its left, the pattern then evolves as

$$\begin{array}{lcllll} n & : & 0{*}{*} & 01{*}{*} & 11{*}{*} & {*}1{*}{*} \\ n+1 & : & {*}{*}{*} & {*}{*}{*} & 0{*}{*} & {*}{*}{*}{*} \\ n+2 & : & & & {*}{*}{*} & \end{array} \qquad (8)$$

We now consider the implications of these results. The sequence of * grows in the evolution if it includes **, the asymptotic solution includes neither 0 nor 1 after enough time steps, and $0 < u < 1$ holds for any u at $n \gg 0$. Since this type of asymptotic solution is of type B, we will discuss it in the next section.

Thus, if * is included in the solution of type A, it must be isolated as 0*0, 0*1, 1*0, or 1*1. We now consider the local sequence $u_j^n \dots u_{j+4}^n$ for any combination of 0,

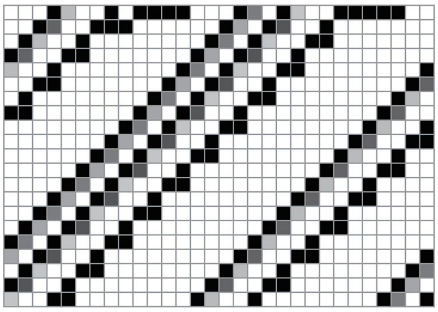

Fig. 3. Example of time evolution that becomes an asymptotic solution of type A.

1, and $*$, which determines the sequence $u^{n+1}_{j+1}u^{n+1}_{j+2}u^{n+1}_{j+3}$ and verifies which sequences can produce an isolated $*$. Using the rule tables (5) and (7), we can show that only the following four patterns are valid:

$$\begin{array}{cccccc} n & : & 001{*}0 & 101{*}0 & {*}01{*}0 & 11{*}11 \\ n+1 & : & 1{*}0 & 1{*}0 & 1{*}0 & 0{*}0 \end{array} \tag{9}$$

Therefore, if isolated $*$s exist in the asymptotic solution, it must be $01{*}0$ moving to $-j$ direction at speed 1. Note that we used Mathematica to derive (9) since the number of cases is large.

We now discuss about the sequence of 1s. Analogous to the above, considering the local sequence $u^n_j \dots u^n_{j+6}$ of any combination of values of 0, 1, and $*$ and calculating $u^{n+2}_{j+2}u^{n+2}_{j+3}u^{n+2}_{j+4}$, we can show that any combination at n cannot produce 111 at $n+2$. Therefore, if a sequence of 1s is included in the asymptotic solution, it must be $x11y$ or $x1y$ where x and y are 0 or $*$. Moreover, considering the isolated $*$ is $01{*}0$ as shown above, the sequence of 1s must be one of the forms 0110, 0100, 0101, or $01{*}0$. Calculating possible local sequences $u^n_j \dots u^n_{j+7}$ that produce $u^{n+2}_{j+2}u^{n+2}_{j+3}u^{n+2}_{j+4}u^{n+2}_{j+5} = 0110, 0100$, or 0101, we obtain the following evolutions:

$$\begin{array}{ccrr} n & : & \dots.0110 & \dots.0100 \\ n+1 & : & ..0100 & ..0110 \\ n+2 & : & 0110 & 0100 \end{array} \tag{10}$$

Note that 0101 cannot be produced and the symbol "." denotes an appropriate value.

Summarizing the above results, about the asymptotic solutions of type A, we have

- If $*$s exist, they are isolated and realized by the local pattern $01{*}0$ moving in the $-j$ direction at speed 1.
- If 1s exist other than $01{*}0$, they are given by 0110 or 0100. These two patterns appear alternately as time proceeds and move in the $-j$ direction at speed 1.
- Among local patterns $1*$, 11, and 10, one or more 0s exist.

An example of time evolution of type A is shown in Fig. 3.

3. Asymptotic solutions with neither 0 nor 1

In this section, we discuss asymptotic solutions of type B, that is, $0 < u^n_j < 1$ for any j for $n \to \infty$. We can prove that any u^n_j converges to $\omega = (3-\sqrt{5})/2$ and the solution becomes uniform with a constant ω as follows.

Assume any pair of a and b satisfying,

$$0 < a \le \omega, \qquad \frac{1-a}{2-a} \le b \le \frac{1-2a}{1-a}. \tag{11}$$

We can derive $(1-a)/(2-a) \le (1-2a)/(1-a)$ if $0 < a \le \omega$, and $a \le b$. Moreover, the minimum and maximum of $f(x,y,z)$ in the range $x, y, z \in [a,b]$ are

$$\begin{aligned} \min_{x,y,z\in[a,b]} f(x,y,z) &= f(b,a,a) = a(1-a)(2-b), \\ \max_{x,y,z\in[a,b]} f(x,y,z) &= f(a,a,b) = (1-a)(a+b-ab). \end{aligned} \tag{12}$$

Note that we can derive the above by comparing the values of $f(x,y,z)$ for $x, y, z \in \{a,b\}$ since $f(x,y,z)$ is linear on x, y, and z respectively.

Next, let us consider the following sequences for a_n and b_n:

$$a_{n+1} = f(b_n,a_n,a_n), \qquad b_{n+1} = f(a_n,a_n,b_n). \tag{13}$$

If we assume

$$0 < a_n \le \omega, \qquad \frac{1-a_n}{2-a_n} \le b_n \le \frac{1-2a_n}{1-a_n}, \tag{14}$$

we can derive $a_n \le a_{n+1}$ and $b_n \ge b_{n+1}$ since we have

$$\begin{aligned} a_{n+1} - a_n &= a_n(1-a_n)\Big(\frac{1-2a_n}{1-a_n} - b_n\Big) \ge 0, \\ b_{n+1} - b_n &= a_n(2-a_n)\Big(\frac{1-a_n}{2-a_n} - b_n\Big) \le 0. \end{aligned} \tag{15}$$

Moreover, $a_{n+1} \le \omega$ holds if (14) is assumed, since

$$\begin{aligned} a_{n+1} - \omega &= a_n(1-a_n)(2-b_n) - \omega \\ &\le a_n(1-a_n)\Big(2 - \frac{1-a_n}{2-a_n}\Big) - \omega \\ &= \frac{(a_n-\omega)(a_n^2-(4-\omega)a_n+2)}{2-a_n} \le 0. \end{aligned} \tag{16}$$

About $b_{n+1} = (1-a_n)(a_n+(1-a_n)b_n)$, we obtain

$$\frac{1-a_n}{2-a_n} \le b_{n+1} \le (1-a_n)^2, \tag{17}$$

if (14) is assumed. The inequalities

$$\frac{1-a_{n+1}}{2-a_{n+1}} \le \frac{1-a_n}{2-a_n}, \qquad (1-a_n)^2 \le \frac{1-2a_{n+1}}{1-a_{n+1}} \tag{18}$$

hold since we can derive

$$\frac{1-a_n}{2-a_n} - \frac{1-a_{n+1}}{2-a_{n+1}} = \frac{a_n(1-a_n)\left(\dfrac{1-2a_n}{1-a_n} - b_n\right)}{(2-a_n)(2-a_{n+1})} \ge 0 \tag{19}$$

and

$$\begin{aligned} &\frac{1-2a_{n+1}}{1-a_{n+1}} - (1-a_n)^2 \\ &\quad = \frac{(1-a_n)(1+2a_n-a_n^2)b_n - a_n(3-6a_n+2a_n^2)}{1-a_{n+1}} \\ &\quad \ge \frac{a_n(1-3a_n+a_n^2)^2}{2-a_n} \ge 0. \end{aligned} \tag{20}$$

Thus, the same form of inequalities as in (14) holds for a_{n+1} and b_{n+1} as

$$0 < a_{n+1} \leq \omega, \quad \frac{1-a_{n+1}}{2-a_{n+1}} \leq b_{n+1} \leq \frac{1-2a_{n+1}}{1-a_{n+1}} \quad (21)$$

under the assumption (14).

Summarizing the above, we arrive at

Proposition 1 *Consider the sequences on a_n and b_n ($n \geq 0$):*

$$a_{n+1} = f(b_n, a_n, a_n), \qquad b_{n+1} = f(a_n, a_n, b_n) \quad (22)$$

with initial terms a_0 and b_0 satisfying

$$0 < a_0 < \omega, \qquad \frac{1-a_0}{2-a_0} \leq b_0 \leq \frac{1-2a_0}{1-a_0}. \quad (23)$$

Then, a_n and b_n for any n satisfy the same form of inequalities as of a_0 and b_0. Moreover, $a_n \leq a_{n+1}$ and $b_{n+1} \leq b_n$ hold and the interval $[a_n, b_n]$ is nested as $[a_{n+1}, b_{n+1}] \subseteq [a_n, b_n]$.

Since the sequence of intervals $[a_n, b_n]$ is nested, it converges to $[\alpha,\beta]$ for $n \to \infty$. Values α and β satisfy $\alpha = f(\beta, \alpha, \alpha)$ and $\beta = f(\alpha, \alpha, \beta)$. The solution of these two equations is uniquely determined as $\alpha = \beta = \omega$ in the range $0 < \alpha \leq \beta < 1$.

Finally, let us consider the asymptotic solution of type B. Without loss of generality, we can assume that the time step of the asymptotic solution is $n = 0$. The solution satisfies $0 < u_j^0 < 1$ for any j. The size of space sites is finite ($0 \leq j < K$) and a periodic boundary condition is imposed. Since the space is finite, there exist a maximum M and minimum m for $\{u_j^n\}_{j=0}^{K-1}$. Then, we can choose a_0 satisfying $0 < a_0 < \min(m, \omega)$. The upper bound $(1-2a_0)/(1-a_0)$ for b_0 of Proposition 1 converges to 1 as $a_0 \to 0$, while the lower bound $(1-a_0)/(2-a_0)$ converges to $1/2$. Therefore, we can always choose a_0 and b_0 satisfying the inequalities of Proposition 1 for any initial data. Since a_{n+1} and b_{n+1} are the maximum and the minimum of $f(x, y, z)$ in the range of x, y, $z \in [a_n, b_n]$, $u_j^n \in [a_n, b_n]$ holds for any n from Proposition 1. Thus, we obtain $\lim_{n\to\infty} u_j^n = \omega$ for any j. This implies that the asymptotic solution of type B is a uniform solution with the value ω.

4. Concluding remarks

We have discussed the asymptotic solutions to (4). The solutions are always classified into two types, namely, type A and B. We have shown that the stable propagating wave with local pattern 0110, 0100, or 01∗0 exists in type A. Binary solutions constructed only from $\{0, 1\}$ are classified into this type. On the other hand, the asymptotic solution of type B is the unique uniform solution $u \equiv \omega$.

Although the asymptotic solutions are completely classified, it is more difficult to solve the initial value problem of (4). We leave it to future work. Moreover, similar results to type B have been reported for other fuzzy CAs [9, 10]. Analyzing these equations and classifying their solutions is another direction for future research.

There are various forms of fuzzy CA produced from their original CA. We have also studied another example of fuzzy CA obtained by replacing the term xyz by xy^2z in (4) and confirmed through numerical calculations that the asymptotic solutions are classified into two types similar to (4). It is also a future problem to develop a general method to classify fuzzy CAs originating from the same CA according to their behavior of solutions.

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