Max-min-plus expressions for one-dimensional particle cellular automata obtained from a fundamental diagram

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Abstract. We study one-dimensional neighborhood-five conservative cellular automata (CA), referred to as particle cellular automata five (PCA5). We show that evolution equations for PCA5s that belong to certain types can be obtained in the form of max-min-plus expressions from a fundamental diagram. The obtained equations are transformed into other max-min-plus expressions by ultradiscrete Cole-Hopf transformation, which enable us to analyze the asymptotic behaviors of general solutions. The equations in the Lagrange representation, which describe particle motion, are also presented, which too can be obtained from a fundamental diagram. Finally, we discuss the generalization to a one-dimensional conservative neighborhood-n CA, i.e., PCAn.

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1. Introduction

Cellular automata (CA) are dynamic systems in which space and time are discrete and physical quantities take on a finite set of discrete values. Despite their simple construction, CA exhibit complicated behavior and generate complex patterns. By virtue of these properties, CA have been used as mathematical models for complex phenomena in the fields of physics, chemistry, biology, economics, and sociology[1, 2].

Recently, one-dimensional conservative CA have attracted much attention as models for traffic flow[3]. One of the simplest models is Rule 184, which is a neighborhood-three CA (also called an elementary CA). Since Rule 184 exhibits phase transition from a free-flow state to a congestion state as real traffic flow does, it has been used as a basic model and extended to many other CA models. However, despite the intensive research on CA models for traffic flow, there are few studies dealing with all one-dimensional conservative CA in a unified way.

A powerful method for dealing with CA is the ultradiscretization method[4, 5], which connects difference equations and CA by applying the following simple formula for the transformation of variables:

$$\lim_{\varepsilon\to+0} \varepsilon \log\left(e^{A/\varepsilon} + e^{B/\varepsilon}\right) = \max(A, B). \tag{1}$$

A conservative CA called the "box and ball system" (BBS) is obtained from the discrete KdV equation using this method, and other CA that possess a solitonic nature are obtained from soliton equations. Moreover, Rule 184 is derived from the discrete Burgers equation[6].

Equation (1) shows that the addition of the difference equations corresponds to the max operation of the ultradiscrete equation. The multiplication and division correspond to $+$ and $-$, respectively, since we have

$$\varepsilon \log\left(e^{A/\varepsilon} \times e^{B/\varepsilon}\right) = A + B, \qquad \varepsilon \log\left(e^{A/\varepsilon} / e^{B/\varepsilon}\right) = A - B$$

Though the subtraction is not well defined for the ultradiscrete equation, we can automatically obtain the ultradiscrete equation and its solution by replacing $+$, $\times$ and $/$ by max, $+$ and $-$, respectively, if the subtraction is not included explicitly in the difference equations and their solutions. The algebra defined by operations of max, $+$, and $-$ is called "max-plus" algebra. Binary operations such as AND, OR, and NOT can be easily converted to max-plus operations.

In a previous study[7], we investigated one-dimensional neighborhood-four conservative CA, referred to as particle cellular automata four (PCA4), and found that evolution equations for PCA4-1, 4-2, and 4-3 can be obtained in the form of max-plus expressions in combination with "min," i.e., "max-min-plus" expressions, which are the key to solving the equations and analyzing asymptotic behaviors.

In this study, we investigate PCA5s and show that the evolution equations for PCA5s that belong to certain types can be obtained in the form of max-min-plus expressions from fundamental diagrams. In addition, we show that the equations obtained are transformed into other max-min-plus expressions by ultradiscrete Cole-Hopf transformation, which enables us to analyze the asymptotic behaviors of general solutions. Furthermore, we give the Lagrange representation for the class of PCA5s, which describes the motion of particles and can be obtained from a fundamental diagram. Finally, we discuss the generalization to the neighborhood-n CA case, that is, PCAn.

2. Particle CA

Let us consider the following equation

$$u_j^{n+1} = f(u_{j-l}^n, u_{j-l+1}^n, \ldots, u_{j+r}^n) \qquad (-\infty < j < \infty, 0 \leq n), \tag{2}$$

where n and j are integers representing the timestep and space site number, respectively, and l and r are positive integer constants. Assuming that u takes the value of 0 or 1, (2) is an evolution equation for a one-dimensional CA with the rule defined by $R(= l+r+1)$ variable function f. We call this CA a "neighborhood-R CA." Following Wolfram[1, 2], to each Rule f, we assign a rule number $N(f)$ such that

$$N(f) = \sum_{(u_1,u_2,\ldots,u_R)\in\{0,1\}^R} f(u_1, u_2, \ldots, u_R) 2^{2^{R-1}u_1+2^{R-2}u_2+\ldots+2^0 u_R}.$$

In this study, we focus on the one-dimensional CAs that satisfy

$$\sum_{j=1}^{K} u_j = \sum_{j=1}^{K} f(u_{j-l}, u_{j-l+1}, \ldots, u_{j+r}) \tag{3}$$

where $K(\geq R)$ is the size of the one-dimensional lattices, and $u_{j+K} = u_j$ holds for all j. From (3), the condition

$$\sum_{j=1}^{K} u_j^n = \sum_{j=1}^{K} u_j^{n+1}, \tag{4}$$

holds, which means that the sum of u, that is the number of 1s at all space sites, is conserved for arbitrary n. Let u_j^n denote the number of particles at the jth site and the nth timestep, and each "1" in the solution represents a particle. Then, particles move among sites according to the evolution rule defined by (2) without creation or annihilation. We call the CA satisfying this condition (4) a "particle CA" (PCA) in this sense.

3. PCA5

In [8], Hattori and Takesue showed that a neighborhood-R CA with Rule f is a PCA if and only if f satisfies

$$f(u_1, u_2, \ldots, u_R) - u_{l+1} = q(u_1, u_2, \ldots, u_{R-1}) - q(u_2, u_3, \ldots, u_R), \tag{5}$$

$$q(u_1, u_2, \ldots, u_{R-1}) = \sum_{k=1}^{l} u_k - \sum_{k=1}^{R-1} f(\underbrace{0, 0, \ldots, 0}_{k}, u_1, u_2, \ldots, u_{R-k}), \tag{6}$$

for all $(u_1, u_2, \ldots, u_R) \in \{0,1\}^R$.

Applying Hattori and Takesue's result to neighborhood-five CA expressed by the equation

$$u_j^{n+1} = f(u_{j-2}^n, u_{j-1}^n, u_j^n, u_{j+1}^n, u_{j+2}^n),$$

we obtain a system of 2^5 linear equations with 2^5 variables $f(a,b,c,d,e)$ for all $(a,b,c,d,e) \in \{0,1\}^5$ from (5) and (6). By finding binary solutions to the system, we obtain 428 rules for one-dimensional conservative neighborhood-five CA (PCA5). Reflection symmetry

$$f_1(a,b,c,d,e) = f_2(e,d,c,b,a)$$

and Boolean conjugation symmetry

$$f_1(a,b,c,d,e) = 1 - f_2(1-a, 1-b, 1-c, 1-d, 1-e)$$

for rules f_1 and f_2 reduce the 428 rules to 129 rules. Furthermore, by excluding the rules for neighborhood-three and -four CA (PCA3 and PCA4, respectively), we have the following 115 rules:

2863377064, 2881005752, 2881267852, 2881398914, 2881464448, 2944969912,
2945232012, 2945363074, 2945428608, 2947326124, 2947457186, 3098065832,
3099375756, 3099506818, 3099572352, 3102247912, 3103295736, 3103557836,
3103688898, 3103754432, 3116153216, 3120335296, 3132144268, 3132275330,
3136326348, 3136457410, 3136522944, 3137047048, 3148921728, 3153103808,
3153627912, 3163077816, 3163470978, 3163536512, 3165434028, 3165630624,
3167259896, 3167521996, 3167653058, 3167718592, 3169616108, 3169812704,
3180117376, 3182211488, 3184299456, 3186393568, 3196108428, 3198202540,
3200290508, 3200487104, 3201011208, 3202384620, 3202581216, 3203105320,
3213933712, 3214980000, 3216027824, 3216289924, 3217067968, 3217592072,
3218115792, 3218639896, 3219162080, 3219686184, 3220209904, 3220472004,
3220734008, 3220996108, 3221127170, 3366517672, 3367565496, 3367827596,
3370699752, 3372009676, 3384605056, 3388787136, 3400596108, 3404778188,
3421555648, 3422079752, 3431529656, 3431791756, 3435711736, 3448569216,
3450663328, 3452751296, 3454845408, 3464560268, 3482385552, 3484479664,
3484741764, 3486043912, 3486567632, 3487091736, 3487613920, 3488138024,
3488923844, 3489185848, 3639663552, 3640187656, 3703627712, 3704151816,
3705199640, 3705721824, 3706245928, 3706769648, 3707031748, 3707293752,
3771264248, 3822120144, 3822644248, 3824214256, 3824738360, 3888178416,
4040228048,

which is the smallest rule number among their equivalent rules. In this paper, we assign a number "m" in the range 1-115 to each rule, and call them PCA5-m.

4. Obtaining evolution equations in the form of max-min-plus expressions from a fundamental diagram

In the field of traffic-flow theory, a fundamental diagram is a useful tool for determining the traffic state of a roadway. This diagram relates traffic flux and traffic density. Under periodic boundary conditions, the density is defined by

$$\rho = \frac{1}{K} \sum_{j=1}^{K} u_j^n,$$

where K is the period of sites. Since the number of particles is conserved, ρ is a constant irrespective of time n. The average flux is defined by

$$\bar{q}^n = \frac{1}{K} \sum_{j=1}^{K} q(u_j^n, \ldots, u_{j+R-1}^n).$$

For large enough n, the evolution often approaches a steady state where $\bar{q}^n$ becomes a constant. In case of convergence, the constant is defined by

$$Q = \lim_{n\to\infty} \bar{q}^n.$$

The graph of Q versus ρ is called a "fundamental diagram."

The following subsection shows that the evolution equations for PCA5s that belong to certain types can be obtained in the form of max-min-plus expressions from a fundamental diagram.

4.1. Type-A

Let us take PCA5-34 (Rule 3163536512) as an example. The evolution equation for PCA5-34 is given by

$$u_j^{n+1} = f(u_{j-2}^n, u_{j-1}^n, u_j^n, u_{j+1}^n, u_{j+2}^n) \tag{7}$$

together with the following rule table of f.

$abcde$	11111	11110	11101	11100	11011	11010	11001	11000	10111	10110	10101	10100	10011	10010	10001	10000
$f(a,b,c,d,e)$	1	0	1	1	1	1	0	0	1	0	0	0	1	1	1	1
$abcde$	01111	01110	01101	01100	01011	01010	01001	01000	00111	00110	00101	00100	00011	00010	00001	00000
$f(a,b,c,d,e)$	1	0	1	1	1	1	0	0	1	0	0	0	0	0	0	0

The upper row of the table shows all possible combinations of binary arguments, and the lower row gives the value of f.

From (5) and (6), the evolution equation for PCA5-34 can be rewritten as

$$u_j^{n+1} = u_j^n + q(u_{j-2}^n, u_{j-1}^n, u_j^n, u_{j+1}^n) - q(u_{j-1}^n, u_j^n, u_{j+1}^n, u_{j+2}^n) \tag{8}$$

together with the following rule table of q.

$abcd$	1111	1110	1101	1100	1011	1010	1001	1000	0111	0110	0101	0100	0011	0010	0001	0000
$q(a,b,c,d)$	0	1	1	1	0	0	1	1	0	1	1	1	0	0	0	0

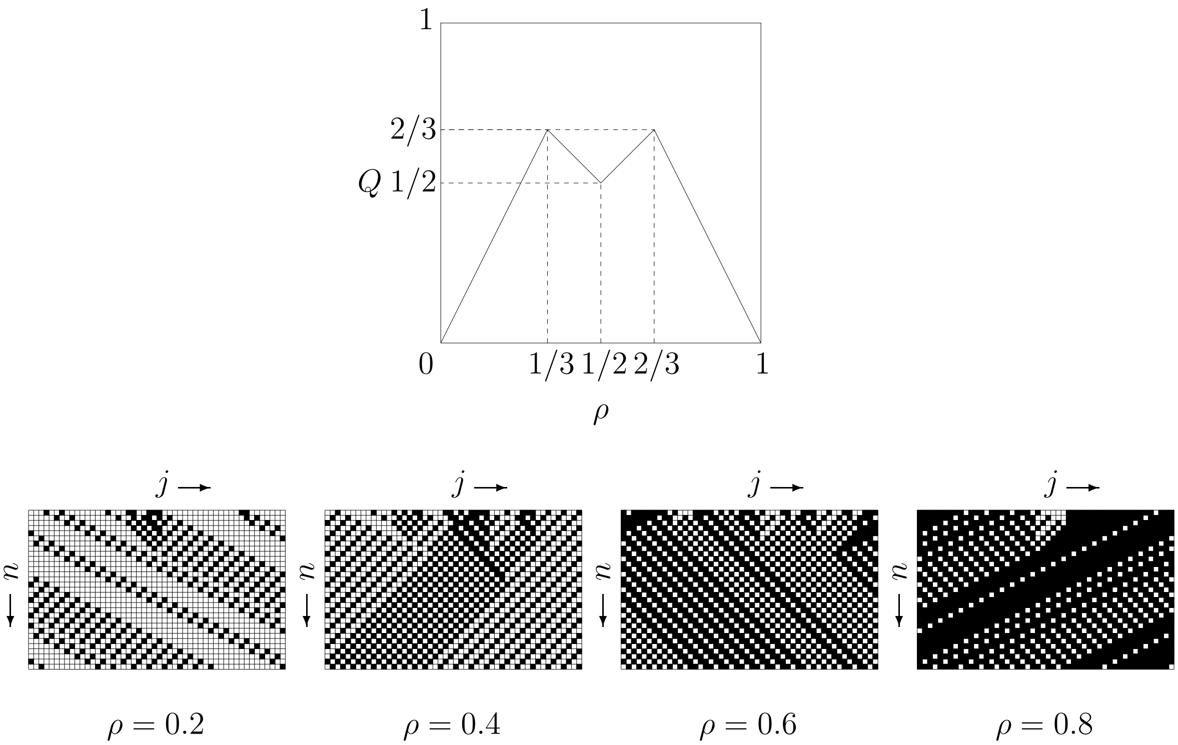

Figure 1. Fundamental diagram and space-time evolutions of PCA5-34

The upper row of the table shows all possible combinations of binary arguments, and the lower row gives the value of q.

Employing numerical simulation for (8), we obtain the fundamental diagram and examples of evolutions for PCA5-34 shown in Figure 1.

Since the fundamental diagram in Figure 1 is a piecewise linear curve, the relationship between Q and ρ can be expressed in terms of the max-min-plus expression,

$$Q(\rho) = \max(\min(2\rho, 1-\rho), \min(\rho, 2-2\rho)), \tag{9}$$

which is a piecewise linear function composed of four linear functions 2ρ, $1-\rho$, ρ, and $2-2\rho$. (See Figure 2.)

Let us consider replacing variables $m\rho$ in (9) by

$$m\rho \to \begin{cases} \displaystyle\sum_{k=1}^{m} u_{j-k}^n & (m>0) \\ \displaystyle -\sum_{k=1}^{-m} u_{j+k-1}^n & (m<0) \end{cases}. \tag{10}$$

Then, we obtain

$$Q(u_{j-2}^n, u_{j-1}^n, u_j^n, u_{j+1}^n) = \max(\min(u_{j-2}^n + u_{j-1}^n, 1-u_j^n), \min(u_{j-1}^n, 2-u_j^n-u_{j+1}^n)) \tag{11}$$

from (9). (See Figure 3.)

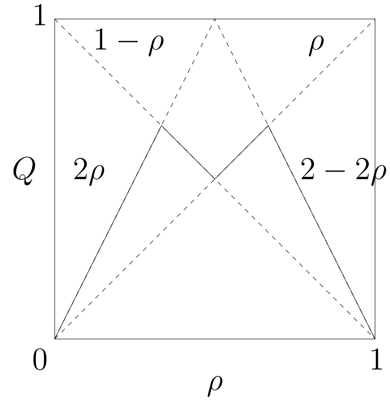

Figure 2. Four linear functions of ρ and the composed piecewise linear function

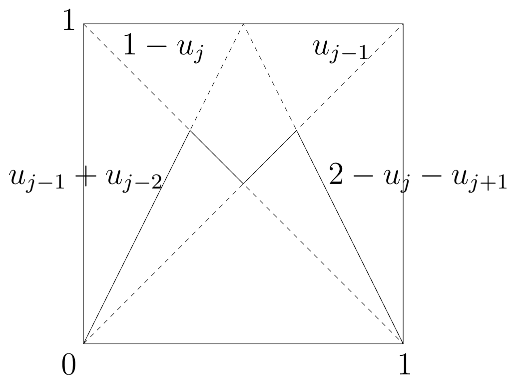

Figure 3. Obtaining expression for flux from the fundamental diagram

Since $Q(u_{j-2}^n, u_{j-1}^n, u_j^n, u_{j+1}^n)$ takes exactly same values as those of the rule table for $q(u_{j-2}^n, u_{j-1}^n, u_j^n, u_{j+1}^n)$ of (8) for all possible combinations of binary values, (8) can be rewritten as

$$\begin{aligned} u_j^{n+1} = u_j^n &+ \max(\min(u_{j-2}^n + u_{j-1}^n, 1 - u_j^n), \min(u_{j-1}^n, 2 - u_j^n - u_{j+1}^n)) \\ &- \max(\min(u_{j-1}^n + u_j^n, 1 - u_{j+1}^n), \min(u_j^n, 2 - u_{j+1}^n - u_{j+2}^n)). \end{aligned} \qquad (12)$$

This is the max-min-plus expression for PCA5-34. Introducing the ultradiscrete Cole-Hopf transformation from u to F,

$$u_j^n = F_j^n - F_{j-1}^n, \qquad (13)$$

we obtain the evolution equation in the form of max-min-plus expressions for F,

$$F_j^{n+1} = \min(\max(F_{j-2}^n, F_{j+1}^n - 1), \max(F_{j-1}^n, F_{j+2}^n - 2)), \qquad (14)$$

which is composed of linear functions of $F_{j+k}^n\,(k \in \mathbb{Z})$. Being able to express evolution equations for F as a combination of linear functions of F is critical for analyzing the asymptotic behavior of general solutions of PCA. We used this fact for the analysis of the asymptotic behavior of PCA4 in a previous study[7].

Note here that (14) can be directly obtained from (9) by applying the following replacements. (See Figure 4.)

$$\begin{aligned} m\rho + a &\to F^n_{j-m} - a \\ \max &\to \min \\ \min &\to \max \end{aligned}$$

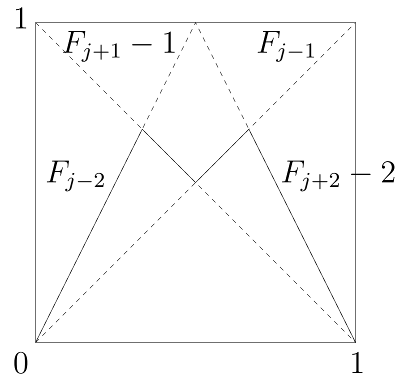

Figure 4. Obtaining equations for F from the fundamental diagram

It is known that PCAs allow two different representations: *Euler* representation and *Lagrange* representation. In the Euler representation, particles are observed at a certain fixed point in space as dependent(field) variables, while in the Lagrange representation, we trace each particle and follow its trajectory. Thus, a dependent variable represents each particle's position in the Lagrange representation.

In previous studies[9, 10], we proposed an Euler-Lagrange transformation for PCAs by developing the following transformation formulas for the variable of Euler representation u^n_j, which denotes the number of particles at the jth site and nth timestep, and the variable of Lagrange representation x^n_i, which denotes the position of the ith particle at the nth timestep,

$$u^n_j = F^n_j - F^n_{j-1}, \tag{15}$$

$$F^n_j = \sum_{i=1}^{N} H(j - x^n_i), \tag{16}$$

where $H(x)$ is the step function defined by $H(x) = 1$ if $x \geq 0$, and $H(x) = 0$ otherwise. Applying the transformation (16) to (14) and using the formulas,

$$\sum_{i=1}^{N} H(j - \min(a_i, b_i)) = \max(\sum_{k=1}^{N} H(j - a_i), \sum_{k=1}^{N} H(j - b_i)), \tag{17}$$

$$\sum_{i=1}^{N} H(j - \max(a_i, b_i)) = \min(\sum_{k=1}^{N} H(j - a_i), \sum_{k=1}^{N} H(j - b_i)), \tag{18}$$

$$\max(\sum_i H(j-a_i)-m,0)=\sum_i H(j-a_{i+m}), \tag{19}$$

where we assume that $a_1 < a_2 < \cdots < a_N$ and $b_1 < b_2 < \cdots < b_N$, we obtain

$$x_i^{n+1} = \max(\min(x_i^n+2, x_{i+1}^n-1), \min(x_i^n+1, x_{i+2}^n-2)), \tag{20}$$

which is the Lagrange representation for PCA5-34 in the form of a max-min-plus expression. Note here that (20) can be directly obtained from (9) by applying the following replacements. (See Figure 5.)

$$m\rho + a \to x_{i+a} + m \tag{21}$$

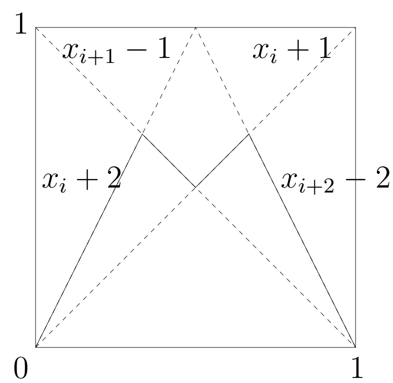

Figure 5. Obtaining equations for x from the fundamental diagram

To summarize, the procedure we have performed above is as follows.

(i) Employ numerical simulation for PCA and obtain the fundamental diagram.

(ii) If the fundamental diagram is a piecewise linear curve, construct $Q(\rho)$ from the fundamental diagram.

(iii) Construct flux q from $Q(\rho)$ and obtain the evolution equation (Euler representation),

$$u_j^{n+1} = u_j^n + q(u_{j-2}^n, u_{j-1}^n, u_j^n, u_{j+1}^n) - q(u_{j-1}^n, u_j^n, u_{j+1}^n, u_{j+2}^n).$$

(iv) Obtain the evolution equation for F,

$$F_j^{n+1} = \phi(F_{j-2}^n, F_{j-1}^n, F_j^n, F_{j+1}^n, F_{j+2}^n).$$

(v) Obtain the evolution equation for x (Lagrange representation),

$$x_i^{n+1} = p(x_{j-2}, x_{j-1}, x_j, x_{j+1}, x_{j+2}).$$

Among the 115 rules for PCA5, there are 17 for which the evolution equations for u_j^n, F_j^n, and x_i^n can be obtained in the form of max-min-plus expressions from fundamental diagrams by using the procedure described above. These equations are given in Appendix A. In the tables of the Appendix, m denotes the number of PCA5-m and $N(f)$ denotes the rule number defined by Wolfram. Hereafter, we call the 17 rules type-A.

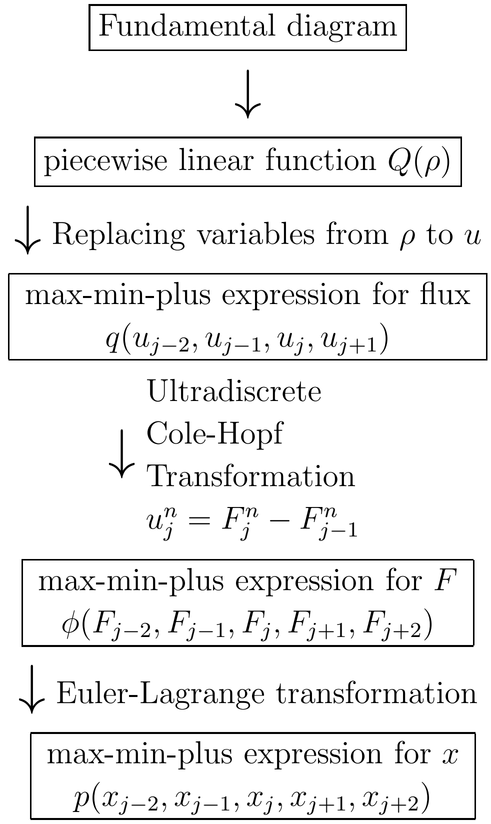

Figure 6. Procedure for obtaining evolution equations of type-A PCA5 in max-min-plus expression

4.2. Type-B

Other than type-A, there exist PCA5 rules for which the fundamental diagram is a piecewise linear curve, and the evolution equations for u_j^n, F_j^n, and x_i^n can be obtained.

Let us consider PCA5-15 (rule 3099572352). Employing numerical simulation, we obtain the fundamental diagram for PCA5-15, as shown in figure 7. figure 7 indicates that the piecewise linear curve can be composed from two linear functions 2ρ and $2-2\rho$ as follows.

$$Q(\rho) = \min(2\rho, 2-2\rho) \tag{22}$$

Following the same procedure as in the case of type-A, we obtain a flux as follows:

$$q(u_{j-2}, u_{j-1}, u_j, u_{j+1}) = \min(u_{j-2} + u_{j-1}, 2 - u_j - u_{j+1}) \tag{23}$$

Evaluating the values of (23) for all possible combinations of binary values, we obtain the following table.

$abcd$	1111	1110	1101	1100	1011	1010	1001	1000	0111	0110	0101	0100	0011	0010	0001	0000
$q(a,b,c,d)$	0	1	1	2	0	1	1	1	0	1	1	1	0	0	0	0

However, the flux of PCA5-15 is given by

$abcd$	1111	1110	1101	1100	1011	1010	1001	1000	0111	0110	0101	0100	0011	0010	0001	0000
$q(a,b,c,d)$	0	1	1	1	0	1	1	1	0	1	1	1	0	0	0	0

and the value of $q(1,1,0,0)$ is found to be different. Thus, (23) is not the flux of PCA5-15.

To solve this problem, let us suppose that the piecewise linear curve in Figure 7 is not composed of two but more than two linear functions. Let us consider

$$Q(\rho) = \min(1, 2\rho, 2-2\rho) \tag{24}$$

instead of (22). Note here that (22) and (24) are identical functions for real values of ρ. (See Figure 8.)

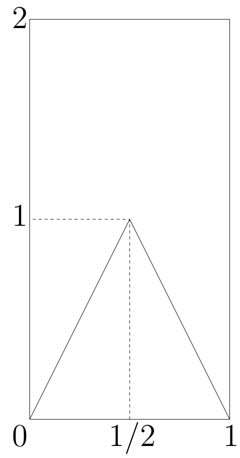

Figure 7. Fundamental diagram of PCA5-15

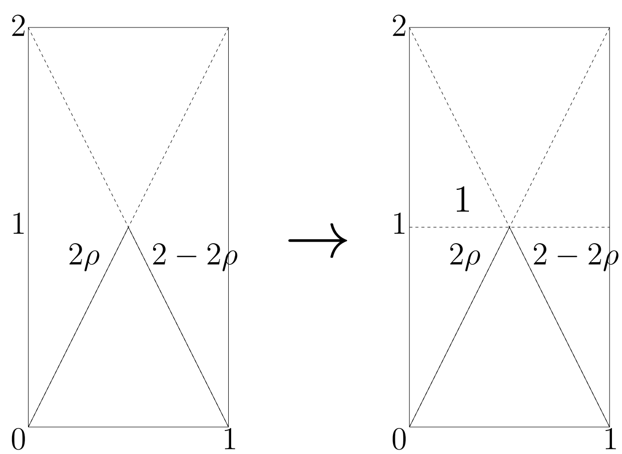

Figure 8. $\min(2\rho, 2-2\rho)$ and $\min(1, 2\rho, 2-2\rho)$

If we apply variable replacement from ρ to u to the above expression, we obtain

$$q(u_{j-2}, u_{j-1}, u_j, u_{j+1}) = \min(1, u_{j-2} + u_{j-1}, 2 - u_j - u_{j+1}) \tag{25}$$

which gives the following table:

abcd	1111	1110	1101	1100	1011	1010	1001	1000	0111	0110	0101	0100	0011	0010	0001	0000
$q(a,b,c,d)$	0	1	1	1	0	1	1	1	0	1	1	1	0	0	0	0

which is identical to the flux of PCA5-15. Thus, the evolution equation in the form of a max-min-plus expression for PCA5-15 is

$$q_j^{n+1} = q_j^n + \min(1, u_{j-2}^n + u_{j-1}^n, 2 - u_j^n - u_{j+1}^n) - \min(1, u_{j-1}^n + u_j^n, 2 - u_{j+1}^n - u_{j+2}^n). \quad (26)$$

We can also obtain the evolution equations for F and x as follows.

$$F_j^{n+1} = \max(F_j^n - 1, F_{j-2}^n, F_{j+2}^n - 2) \quad (27)$$

$$x_i^{n+1} = \min(x_{i+1}^n, x_i^n + 2, x_{i+2}^n - 2) = x_i^n + \min(2, x_{i+1}^n - x_i^n, x_{i+2}^n - x_i^n - 2) \quad (28)$$

There are nine rules for PCA5, of which the evolution equations for u, F, and x are obtained using the procedure described above. These equations are given in Appendix B. We call the nine rules type-B.

4.3. PCA5 rules other than type-A and type-B

We have employed numerical simulations for all 115 PCA5 rules to obtain fundamental diagrams, from which we have obtained evolution equations in the form of max-min-plus expressions for 17 rules of type-A and nine rules of type-B. There are 89 rules for PCA5, for whom evolution equations have not yet been obtained. From the results of numerical simulations, the 89 rules are classified into the following two cases.

- Although the obtained fundamental diagram is a piecewise linear curve, replacing variables from ρ to u does not give the correct flux expression for q. In addition, the type-B procedure does not seem to work well.
- As the obtained fundamental diagram is not a piecewise linear curve, we cannot use the procedure of replacing variables from ρ to u.

At this point, it is not clear if there may be other procedures for obtaining max-min-plus expressions for them.

5. PCAn

We can extend our analysis in the previous section to neighborhood-n CA, i.e., the PCAn case.

Let us consider the case of $n = 6$. One example of PCA6 is Rule 13755053124876288240. From numerical simulation, we obtain the fundamental diagram for the rule shown in Figure 9. Following the procedure of type-A of PCA5, we obtain

$$q(u_{j-2}, u_{j-1}, u_j, u_{j+1}, u_{j+2}) = \max(-u_j, \min(u_{j-2} + u_{j-1} - 1, 1 - u_j - u_{j+1} - u_{j+2})), \\ \min(u_{j-1} - 1, 1 - u_j - u_{j+1}, \min(u_{j-2} + u_{j-1} - 2, 2 - u_j - u_{j+1} - u_{j+2})), \quad (29)$$

$$F_j^{n+1} = \min(F_{j+1}^n, \max(F_{j-2}^n + 1, F_{j+3}^n - 1), \max(F_{j-1}^n + 1, F_{j+2}^n - 1), \max(F_{j-2}^n + 2, F_{j+3}^n - 2)), \quad (30)$$

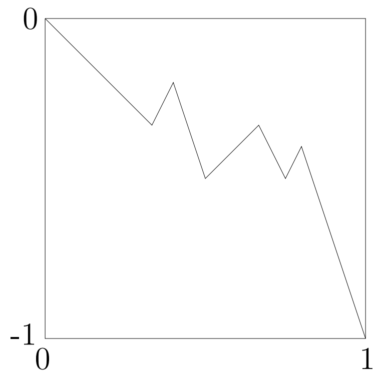

Figure 9. Fundamental diagram for Rule 13755053124876288240

$$x_i^{n+1} = \max(x_i^n - 1, \min(x_{i-1}^n + 2, x_{i+1}^n - 3), \min(x_{i-1}^n + 1, x_{i+1}^n - 2), \min(x_{i-2}^n + 2, x_{i+2}^n - 3)). \tag{31}$$

Although we have not investigated all PCA6 rules, the analysis of PCA5 in the previous section and the example of PCA6 above assure us that our procedure can be applied to PCAn rules.

6. Summary

We have studied PCA5 and shown that the evolution equations for type-A and type-B can be obtained in the form of max-min-plus expressions from a fundamental diagram. The obtained equations have been transformed into max-min-plus expressions that are composed of linear functions of F by ultradiscrete Cole-Hopf transformation. Furthermore, we have obtained the Lagrange representation of the evolution equations.

Although we have not obtained evolution equations for all 115 PCA5 rules, it is important for us to have been able to introduce a unified approach using max-plus algebra, together with the fundamental diagram, for examining PCAs.

In a previous study[7], we analyzed asymptotic behaviors of solutions and derived functions $Q(\rho)$ for PCA4 mathematically. We have not done this for PCA5, but will be able to do so starting from the equations for F obtained in this study. This problem will be addressed in a future study.

Finally, investigating the generalization to the neighborhood-n case is important for the future, and we will report on this in a forthcoming paper.

Appendix A. Type-A

m	$N(f)$	$q(u_{j-2}, u_{j-1}, u_j, u_{j+1})$
1	2863377064	$\min(u_{j-2} + u_{j-1}, 1 - u_j - u_{j+1})$
3	2881267852	$\max(-u_j, \min(u_{j-2} + u_{j-1} - 1, 1 - u_j - u_{j+1}))$
4	2881398914	$\max(-u_j - u_{j+1}, \min(u_{j-2} + u_{j-1} - 1, 1 - u_j - u_{j+1}))$
5	2881464448	$\min(u_{j-2} + u_{j-1}, 2 - u_j - u_{j+1})$
33	3163470978	$\max(-u_j - u_{j+1}, \min(u_{j-2} + u_{j-1} - 1, -u_j), \min(u_{j-1} - 1, 1 - u_j - u_{j+1}))$
34	3163536512	$\max(\min(u_{j-2} + u_{j-1}, 1 - u_j), \min(u_{j-1}, 2 - u_j - u_{j+1}))$
38	3167521996	$\max(-u_j, \min(u_{j-2} + u_{j-1} - 2, 1 - u_j - u_{j+1}))$
39	3167653058	$\max(-u_j - u_{j+1}, \min(u_{j-2} + u_{j-1} - 1, -u_j), \min(u_{j-2} + u_{j-1} - 2, 1 - u_j - u_{j+1}))$
40	3167718592	$\max(\min(u_{j-2} + u_{j-1}, 1 - u_j), \min(u_{j-2} + u_{j-1} - 1, 2 - u_j - u_{j+1}))$
50	3200487104	$\max(\min(u_{j-2} + u_{j-1}, 1 - u_j - u_{j+1}), \min(u_{j-1}, 1 - u_j), \min(u_{j-2} + u_{j-1} - 1, 2 - u_j - u_{j+1}))$
54	3203105320	$\max(\min(u_{j-2} + u_{j-1}, 1 - u_j - u_{j+1}), \min(u_{j-2} + u_{j-1} - 1, 2 - u_j - u_{j+1}))$
59	3217067968	$\max(\min(u_{j-1}, 1 - u_j), \min(u_{j-2} + u_{j-1} - 1, 2 - u_j - u_{j+1}))$
64	3219686184	$\max(\min(u_{j-1}, 1 - u_j - u_{j+1}), \min(u_{j-2} + u_{j-1} - 1, 2 - u_j - u_{j+1}))$
65	3220209904	$\max(0, \min(u_{j-2} + u_{j-1} - 1, 2 - u_j - u_{j+1}))$
68	3220996108	$\max(-u_j, \min(u_{j-2} + u_{j-1} - 1, 2 - u_j - u_{j+1}))$
69	3221127170	$\max(-u_j - u_{j+1}, \min(u_{j-2} + u_{j-1} - 1, 2 - u_j - u_{j+1}))$
96	3488138024	$\max(\min(u_{j-1}, 1 - u_j^n - u_{j+1}), \min(u_{j-2} + u_{j-1} - 1, 1 - u_j))$

m	$N(f)$	$\phi(F_{j-2}, F_{j-1}, F_j, F_{j+1}, F_{j+2})$
1	2863377064	$\max(F_{j-2}, F_{j+2} - 1)$
3	2881267852	$\min(F_{j+1}, \max(F_{j-2} + 1, F_{j+2} - 1))$
4	2881398914	$\min(F_{j+2}, \max(F_{j-2} + 1, F_{j+2} - 1))$
5	2881464448	$\max(F_{j-2}, F_{j+2} - 2)$
33	3163470978	$\min(F_{j+2}, \max(F_{j-2} + 1, F_{j+1}), \max(F_{j-1} + 1, F_{j+2} - 1))$
34	3163536512	$\min(\max(F_{j-2}, F_{j+1} - 1), \max(F_{j-1}, F_{j+2} - 2))$
38	3167521996	$\min(F_{j+1}, \max(F_{j-2} + 2, F_{j+2} - 1))$
39	3167653058	$\min(F_{j+2}, \max(F_{j-2} + 1, F_{j+1}), \max(F_{j-2} + 2, F_{j+2} - 1))$
40	3167718592	$\min(\max(F_{j-2}, F_{j+1} - 1), \max(F_{j-2} + 1, F_{j+2} - 2))$
50	3200487104	$\min(\max(F_{j-2}, F_{j+2} - 1), \max(F_{j-1}, F_{j+1} - 1), \max(F_{j-2} + 1, F_{j+2} - 2))$
54	3203105320	$\min(\max(F_{j-2}, F_{j+2} - 1), \max(F_{j-2} + 1, F_{j+2} - 2))$
59	3217067968	$\min(\max(F_{j-1}, F_{j+1} - 1), \max(F_{j-2} + 1, F_{j+2} - 2))$
64	3219686184	$\min(\max(F_{j-1}, F_{j+2} - 1), \max(F_{j-2} + 1, F_{j+2} - 2))$
65	3220209904	$\min(F_j, \max(F_{j-2} + 1, F_{j+2} - 2))$
68	3220996108	$\min(F_{j+1}, \max(F_{j-2} + 1, F_{j+2} - 2))$
69	3221127170	$\min(F_{j+2}, \max(F_{j-2} + 1, F_{j+2} - 2))$
96	3488138024	$\min(\max(F_{j-1}, F_{j+2} - 1), \max(F_{j-2} + 1, F_{j+1} - 1))$

m	$N(f)$	$p(x_{i-2}, x_{i-1}, x_i, x_{i+1}, x_{i+2})$
1	2863377064	$\min(x_i + 2, x_{i+1} - 2)$
3	2881267852	$\max(x_i - 1, \min(x_{i-1} + 2, x_{i+1} - 2))$
4	2881398914	$\max(x_i - 2, \min(x_{i-1} + 2, x_{i+1} - 2))$
5	2881464448	$\min(x_i + 2, x_{i+2} - 2)$
33	3163470978	$\max(x_i - 2, \min(x_{i-1} + 2, x_i - 1), \min(x_{i-1} + 1, x_{i+1} - 2))$
34	3163536512	$\max(\min(x_i + 2, x_{i+1} - 1), \min(x_i + 1, x_{i+2} - 2))$
38	3167521996	$\max(x_i - 1, \min(x_{i-2} + 2, x_{i+1} - 2))$
39	3167653058	$\max(x_i - 2, \min(x_{i-1} + 2, x_i - 1), \min(x_{i-2} + 2, x_{i+1} - 2))$
40	3167718592	$\max(\min(x_i + 2, x_{i+1} - 1), \min(x_{i-1} + 2, x_{i+2} - 2))$
50	3200487104	$\max(\min(x_i + 2, x_{i+1} - 2), \min(x_i + 1, x_{i+1} - 1), \min(x_{i-1} + 2, x_{i+2} - 2))$
54	3203105320	$\max(\min(x_i + 2, x_{i+1} - 2), \min(x_{i-1} + 2, x_{i+2} - 2))$
59	3217067968	$\max(\min(x_i + 1, x_{i+1} - 1), \min(x_{i-1} + 2, x_{i+2} - 2))$
64	3219686184	$\max(\min(x_i + 1, x_{i+1} - 2), \min(x_{i-1} + 2, x_{i+2} - 2))$
65	3220209904	$\max(0, \min(x_{i-1} + 2, x_{i+2} - 2))$
68	3220996108	$\max(x_i - 1, \min(x_{i-1} + 2, x_{i+2} - 2))$
69	3221127170	$\max(x_i - 2, \min(x_{i-1} + 2, x_{i+2} - 2))$
96	3488138024	$\max(\min(x_i + 1, x_{i+1} - 2), \min(x_{i-1} + 2, x_{i+1} - 1))$

Appendix B. Type-B

m	$N(f)$	$q(u_{j-2}, u_{j-1}, u_j, u_{j+1})$
2	2881005752	$\min\left(\max\left(0, u_{j-2} + u_{j-1} - 1\right), 1 - u_j - u_{j+1}\right)$
13	3099375756	$\max\left(-u_j, \min\left(0, u_{j-2} + u_{j-1} - 1, 1 - u_j - u_{j+1}\right)\right)$
14	3099506818	$\max\left(-u_j - u_{j+1}, \min\left(0, u_{j-2} + u_{j-1} - 1, 1 - u_j - u_{j+1}\right)\right)$
15	3099572352	$\min\left(1, u_{j-2} + u_{j-1}, 2 - u_j - u_{j+1}\right)$
53	3202581216	$\max\left(\min\left(u_{j-2} + u_{j-1}, \max\left(0, 1 - u_j - u_{j+1}\right)\right), \min\left(u_{j-2} + u_{j-1} - 1, 2 - u_j - u_{j+1}\right)\right)$
63	3219162080	$\max\left(\min\left(u_{j-1}, \max\left(0, 1 - u_j - u_{j+1}\right)\right), \min\left(u_{j-2} + u_{j-1} - 1, 2 - u_j - u_{j+1}\right)\right)$
67	3220734008	$\max\left(\min\left(0, 1 - u_j - u_{j+1}\right), \min\left(u_{j-2} + u_{j-1} - 1, 2 - u_j - u_{j+1}\right)\right)$
95	3487613920	$\max\left(\min\left(u_{j-1}, \max\left(0, 1 - u_j - u_{j+1}\right)\right), \min\left(u_{j-2} + u_{j-1} - 1, 1 - u_j\right)\right)$
98	3489185848	$\max\left(\min\left(0, 1 - u_j - u_{j+1}\right), \min\left(u_{j-2} + u_{j-1} - 1, 1 - u_j\right)\right)$

m	$N(f)$	
2	2881005752	$\max\left(\min\left(F_j, F_{j-2} + 1\right), F_{j+2} - 1\right)$
13	3099375756	$\min\left(F_{j+1}, \max\left(F_j, F_{j-2} + 1, F_{j+2} - 1\right)\right)$
14	3099506818	$\min\left(F_{j+2}, \max\left(F_j, F_{j-2} + 1, F_{j+2} - 1\right)\right)$
15	3099572352	$\min\left(F_j - 1, F_{j-2}, F_{j+2} - 2\right)$
53	3202581216	$\min\left(\max\left(F_{j-2}, \min\left(F_j, F_{j+2} - 1\right)\right), \max\left(F_{j-2} + 1, F_{j+2} - 2\right)\right)$
63	3219162080	$\min\left(\max\left(F_{j-1}, \min\left(F_j, F_{j+2} - 1\right)\right), \max\left(F_{j-2} + 1, F_{j+2} - 2\right)\right)$
67	3220734008	$\min\left(\max\left(F_j, F_{j+2} - 1\right), \max\left(F_{j-2} + 1, F_{j+2} - 2\right)\right)$
95	3487613920	$\min\left(\max\left(F_{j-1}, \min\left(F_j, F_{j+2} - 1\right)\right), \max\left(F_{j-2} + 1, F_{j+1} - 1\right)\right)$
98	3489185848	$\min\left(\max\left(F_j, F_{j+2} - 1\right), \max\left(F_{j-2} + 1, F_{j+1} - 1\right)\right)$

m	$N(f)$	
2	2881005752	$\min(\max(0, x_{i-1}+2), x_{i+1}-2)$
13	3099375756	$\max(x_i-1, \min(0, x_{i-1}+2), x_{i+1}-2)$
14	3099506818	$\max(x_i-2, \min(0, x_{i-1}+2), x_{i+1}-2)$
15	3099572352	$\min(x_{i+1}, x_i+2, x_{i+2}-2)$
53	3202581216	$\max(\min(x_i+2, \max(0, x_{i+1}-2)), \min(x_{i-1}+2, x_{i+2}-2))$
63	3219162080	$\max(\min(x_i+1, \max(0, x_{i+1}-2)), \min(x_{i-1}+2, x_{i+2}-2))$
67	3220734008	$\max(\min(x_i, x_{i+1}-2), \min(x_{i-1}+2, x_{i+2}-2))$
95	3487613920	$\max(\min(x_i+1, \max(x_i, x_{i+1}-2)), \min(x_{i-1}+2, x_{i+1}-1))$
98	3489185848	$\max(\min(x_i, x_{i+1}-2), \min(x_{i-1}+2, x_{i+1}-1))$

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