GEOMETRICAL DYNAMICS OF AN INTEGRABLE PIECEWISE-LINEAR MAPPING

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Abstract. A special type of piecewise-linear mapping is discussed. It is obtained by ultradiscretizing the Quispel–Robert–Thompson system. In a special case of a parameter, it becomes a periodic mapping with a constant period for any initial data. In a general case, it becomes an integrable mapping and a period of solution is constant for each solution orbit. We show a structure of solutions discussing the dynamics in a phase plane from a viewpoint of the integrable system theory.

1. Introduction

There have been many studies using a piecewise-linear mapping in the area of dynamical system theory[1]. The standard form of one dimensional mapping is

$$x_{n+1} = f(x_n)$$

where $f(x)$ is linear in each local region of x. For example, the tent map $f(x) = 2x$ $(0 \le x \le 1/2)$, $2(1-x)$ $(1/2 < x \le 1)$ and the Bernoulli shift $f(x) = 2x$ $(0 \le x \le 1/2)$, $2x - 1$ $(1/2 < x \le 1)$ are often used in the chaotic system theory to explain the typical dynamics of chaos. One of the advantages to study a piecewise-linear mapping is that we can analyze its dynamics exactly utilizing the local linearity.

Recently, piecewise-linear mappings appear together with the ultradiscretizing method in the integrable system theory[2]. For example, consider the discrete Painlevé equation[3],

$$x_{n+1} = (1 + \alpha\lambda^n x_n)/x_{n-1},$$

which is integrable because it has a conserved quantity. If we use a transformation of variable x_n and constants α, λ,

$$x_n = e^{X_n/\varepsilon}, \quad \alpha = e^{A/\varepsilon}, \quad \lambda = e^{L/\varepsilon},$$

and take a limit $\varepsilon \to +0$, we obtain an ultradiscrete Painlevé equation

$$X_{n+1} = \max(0, X_n + A + L) - X_{n-1}. \tag{1}$$

Note that the max function is defined by

$$\max(A, B) = \begin{cases} A & (A \geq B) \\ B & (A < B) \end{cases},$$

and we use the following formula in the derivation,

$$\lim_{\varepsilon \to +0} \varepsilon \log(e^{A/\varepsilon} + e^{B/\varepsilon} + \cdots) = \max(A, B, \cdots).$$

The remarkable features of (1) are (i) It is also integrable, that is, it has a conserved quantity, (ii) X can be discrete, that is, X_n is always integer if A, L and initial values of X_n are all integer.

In this paper, we discuss a structure of solutions to an integrable piecewise-linear mapping from a viewpoint of the integrable system theory. The mapping is obtained by ultradiscretizing an integrable difference system, the Quispel–Robert–Thompson (QRT) system. The general form of the QRT system gives a wide range of difference equations[4]. However, when we ultradiscretize the equations, positivity of solution is necessary. Therefore, we restrict its form to the following special one in this paper,

$$x_{n+1} = \frac{1 + a x_n}{x_n^\sigma x_{n-1}}, \tag{2}$$

where a is a constant and $\sigma = 0$, 1 or 2. Using transformations, $x_n = e^{X_n/\varepsilon}$ and $a = e^{A/\varepsilon}$, and taking a limit $\varepsilon \to +0$, we obtain

$$X_{n+1} = \max(0, X_n + A) - \sigma X_n - X_{n-1}, \tag{3}$$

from the above equation[3]. Note that (3) with $\sigma = 0$ is equivalent to (1) with $L = 0$.

First we consider a case of $A = 0$. We show (3) is linearizable in that case by a transformation of variable and its solutions are obtained in an explicit form. Second we consider a case of $A \neq 0$ and discuss a structure of solutions.

2. Periodic case

In this section, we assume $A = 0$ in (3) (or a is positive definite in (2)). Then we obtain

$$X_{n+1} = \max(0, X_n) - \sigma X_n - X_{n-1}. \tag{4}$$

We can easily show that any solution to this equation is always periodic with a constant period. For example, in the case of $\sigma = 0$, $X_2 \sim X_6$ are expressed by initial values X_0 and X_1 as follows:

$$\begin{aligned} &X_2 = \max(0, X_1) - X_0, \quad X_3 = \max(0, X_0, X_1) - X_0 - X_1, \\ &X_4 = \max(0, X_0) - X_1, \quad X_5 = X_0, \quad X_6 = X_1, \end{aligned} \tag{5}$$

where $\max(A, B, C, \cdots)$ denotes the maximum value among A, B, C, $\cdots$. We use the following formulae on max function in the derivation of the above solution,

$$\begin{aligned} &\max(A, B) = \max(B, A), \\ &\max(A, \max(B, C)) = \max(\max(A, B), C) = \max(A, B, C), \\ &\max(A, B) + X = \max(A + X, B + X). \end{aligned}$$

For example, X_3 is expressed by X_0 and X_1 through the following calculation,

$$\begin{aligned} X_3 &= \max(0, X_2) - X_1 = \max(0, \max(0, X_1) - X_0) - X_1 \\ &= \max(X_0, \max(0, X_1)) - X_0 - X_1 = \max(0, X_0, X_1) - X_0 - X_1. \end{aligned}$$

Since (5) gives $X_5 = X_0$ and $X_6 = X_1$ and (4) is of the second order, any solution from arbitrary X_0 and X_1 other than $X_0 = X_1 = 0$ is always periodic with period 5. The case of $X_0 = X_1 = 0$ is exceptional and X_n is always 0 in that case. Similarly, any solution is periodic with period 7 and 8 in the case of $\sigma = 1$ and 2 respectively.

Equation (4) is derived from (2) through the ultradiscretization. If we assume $\sigma = 0$ and $a = 1$ in (2), solutions to (2) are also periodic with a constant period[5]. We obtain the following pattern of solution,

$$\begin{aligned} &x_2 = \frac{1 + x_1}{x_0}, \qquad x_3 = \frac{1 + (1 + x_1)/x_0}{x_1} = \frac{1 + x_0 + x_1}{x_0 x_1}, \\ &x_4 = \frac{1 + (1 + x_0 + x_1)/x_0 x_1}{(1 + x_1)/x_0} = \frac{(1 + x_0)(1 + x_1)}{(1 + x_1) x_1} = \frac{1 + x_0}{x_1}, \\ &x_5 = \frac{1 + (1 + x_0)/x_1}{(1 + x_0 + x_1)/x_0 x_1} = \frac{(1 + x_0 + x_1) x_0}{1 + x_0 + x_1} = x_0, \\ &x_6 = \frac{1 + x_0}{(1 + x_0)/x_1} = x_1. \end{aligned} \tag{6}$$

Therefore, a solution from any positive x_0 and x_1 is always periodic with period 5. Moreover, every solution in (6) is transformed to that in (5) through the above ultradiscretization. It means that both the difference equation and its solution can be transformed consistently through the ultradiscretization.

3. Linearizability of periodic piecewise-linear mapping

Equation (4) can be rewritten by the following piecewise-linear mapping,

$$\begin{cases} X_{n+1} = Y_n \\ Y_{n+1} = \max(0, Y_n) - \sigma Y_n - X_n \end{cases}. \tag{7}$$

The only nonlinearity of this mapping is the term $\max(0, Y_n)$. Therefore a different type of linear mappings are applied to the upper and the lower half plane in a phase plane (X_n, Y_n),

$$\begin{pmatrix} X_{n+1} \\ Y_{n+1} \end{pmatrix} = \begin{cases} \begin{pmatrix} 0 & 1 \\ -1 & 1-\sigma \end{pmatrix} \begin{pmatrix} X_n \\ Y_n \end{pmatrix} & (Y_n \geq 0) \\ \begin{pmatrix} 0 & 1 \\ -1 & -\sigma \end{pmatrix} \begin{pmatrix} X_n \\ Y_n \end{pmatrix} & (Y_n < 0) \end{cases}.$$

We can easily see the periodicity of this mapping by the following geometric dynamics in a phase plane. In the case of $\sigma = 0$, let us consider a sequence of mappings of a point $P_0(c, 0)$ $(c > 0)$ in the phase plane (X_n, Y_n). Then, we obtain a periodic sequence of points,

$$P_0(c, 0) \to P_1(0, -c) \to P_2(-c, 0) \to P_3(0, c) \to P_4(c, c) \to P_0 \to \cdots .$$

Since the parameter c is an arbitrary positive number, the phase plane is divided into 5 local 'fan' areas as shown in Figure 1. Each area is linearly mapped each other in the following order,

$$\mathrm{I} \to \mathrm{II} \to \mathrm{III} \to \mathrm{IV} \to \mathrm{V} \to \mathrm{I} \to \cdots ,$$

and segments P_jP_{j+1} are mapped as follows,

$$P_0P_1 \to P_1P_2 \to P_2P_3 \to P_3P_4 \to P_4P_0 \to P_0P_1 \to \cdots .$$

Though this mapping is nonlinear, it is equivalent to a linear mapping defined by a rotation by an angle $-2\pi/5$, through a combination of local affine transformations. Figure 2 shows corresponding 5 fan areas mapped by this linear mapping in a phase plane (U_n, V_n).

The transformation from (U_n, V_n) to (X_n, Y_n) is again expressed by the max function as follows,

$$\begin{aligned} X_n = \max\Big(& \sin\frac{2\pi}{5} \cdot U_n + (1 - \cos\frac{2\pi}{5}) \cdot V_n,\ \sin\frac{\pi}{5} \cdot U_n + \cos\frac{\pi}{5} \cdot V_n, \\ & \sin\frac{2\pi}{5} \cdot U_n + \cos\frac{2\pi}{5} \cdot V_n \Big). \end{aligned} \tag{8}$$

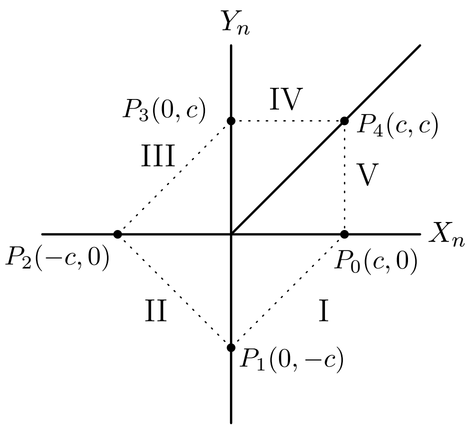

Figure 1. 5 fan areas in the phase plane.

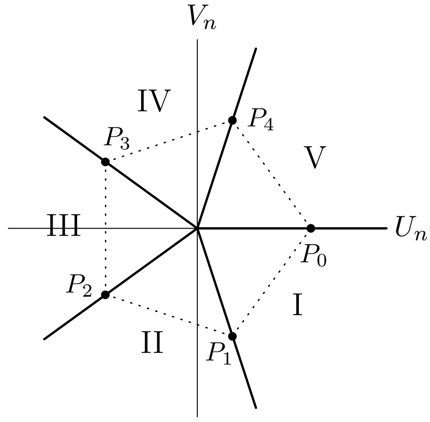

Figure 2. 5 fan areas mapped by a rotation by $-2\pi/5$.

Note that we omit an expression of Y_n since $Y_n = X_{n+1}$. Since $U_n = r_0 \cos(\theta_0 - 2n\pi/5)$ and $V_n = r_0 \sin(\theta_0 - 2n\pi/5)$, we can get a general solution of X_n as follows,

$$X_n = r \cdot \max\Big(-\sin(\theta_0 - \frac{2n+2}{5}\pi) + \sin(\theta_0 - \frac{2n}{5}\pi), \\ -\sin(\theta_0 - \frac{2n+4}{5}\pi),\ \sin(\theta_0 - \frac{2n+8}{5}\pi)\Big),$$

where r (> 0) and θ_0 are arbitrary constants.

We can obtain a general solution of (4) for $\sigma = 1$ and 2 similarly. Thus we show that the mapping (7) equivalent to (4) is a linearizable mapping and solutions are obtained by the linearizability. Note that the mapping, the transformation and the solutions are all expressed by the max function consistently.

4. Integrable case

In the previous sections, we discussed the ultradiscrete QRT system (3) with a special parameter $A = 0$. In this section, we analyze the system with a parameter $A \neq 0$. For simplicity, let us assume $\sigma = 0$. Moreover, if we use a scaling of variable $|A|X_n \to X_n$, then X_n follows

$$X_{n+1} = \max(0, X_n \pm 1) - X_{n-1}, \tag{9}$$

where $X_n + 1$ is chosen when $A > 0$ and $X_n - 1$ when $A < 0$. Therefore, solutions to (3) for $A = \pm 1$ and 0 give those for general A through the scaling. Below we consider only the case of $A = +1$,

$$X_{n+1} = \max(0, X_n + 1) - X_{n-1},$$

or

$$\begin{cases} X_{n+1} = Y_n \\ Y_{n+1} = \max(0, Y_n + 1) - X_n \end{cases}. \tag{10}$$

It is a well known fact that there exists a conserved quantity for (2). In the case of $\sigma = 0$, the quantity is

$$\begin{aligned} h = \frac{1}{x_n x_{n+1}} (a + (1 + a^2)(x_n + x_{n+1}) + a(x_n^2 + x_{n+1}^2) \\ + x_n x_{n+1}(x_n + x_{n+1})). \end{aligned}$$

Using transformations $x_n = e^{X_n/\varepsilon}$ and $a = e^{1/\varepsilon}$ and defining H by $\lim_{\varepsilon \to +0} \varepsilon \log h$, we obtain a conserved quantity for (10),

$$\begin{aligned} H = \max(1 - X_n - Y_n,\ 2 - X_n,\ 2 - Y_n, \\ 1 + X_n - Y_n,\ 1 - X_n + Y_n,\ X_n, Y_n). \end{aligned}$$

Orbits of solutions in the phase plane (X_n, Y_n) are given by contour lines obtained by $H =$ const. Figure 3 shows some contour lines of H. The point $P(1, 1)$ is a fixed point of the mapping, that is, $X_n = Y_n = 1$ for any n if $X_0 = Y_0 = 1$. Positions of vertices of the hexagon Γ are $(3, 3)$, $(3, 1)$, $(1, -1)$, $(-1, -1)$, $(-1, 1)$ and $(1, 3)$.

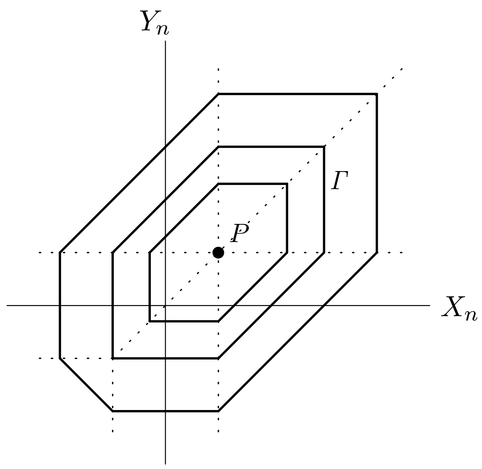

Figure 3. Contour lines of H.

In an inner region of Γ, any solution other than the fixed point P is always periodic with period 6. Since $X_n \geq -1$ and $Y_n \geq -1$ in that region, the mapping (10) becomes a linear mapping and the periodicity is due to this linearity.

In the outer region of Γ, behaviour of solutions becomes more complicated. Solution is still periodic but its period depends on an orbit. Figure 4 shows a solution from $(X_0, Y_0) = (4, 4)$ which is periodic with period 17. Since all segments connecting two neighboring P_j's are included in a region defined by $Y_n \geq -1$ or $Y_n \leq -1$, the segment P_0P_6 is mapped linearly in the following sequence,

$$P_0P_6 \to P_1P_7 \to \cdots \to P_{10}P_{16} \to P_{11}P_0 \to \cdots \to P_{16}P_5 \to P_0P_6 \to \cdots .$$

It means that a solution from any point on the polygon shown in Figure 4 is always periodic with period 17. However, if we change the orbit, the period becomes different. For example, the period of a solution from $(X_0, Y_0) = (5, 5)$ is 11 and that from $(9/2, 9/2)$ is 39.

5. Period of orbit

Next we discuss a relation between the period of a solution to (10) and its orbit. Figure 5 shows a general orbit in the outer region of Γ ($c > 3$). Every point on AC comes back to AC after a certain times of mapping. Any point on AB comes back to AC after 6 mappings and that on BC

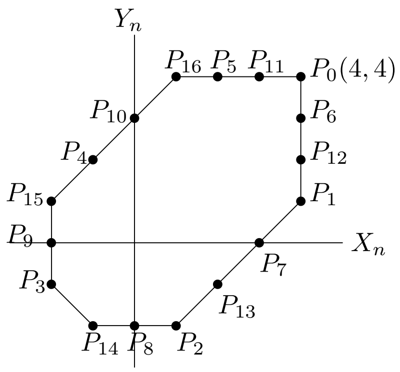

Figure 4. Solution from $(X_0, Y_0) = (4, 4)$

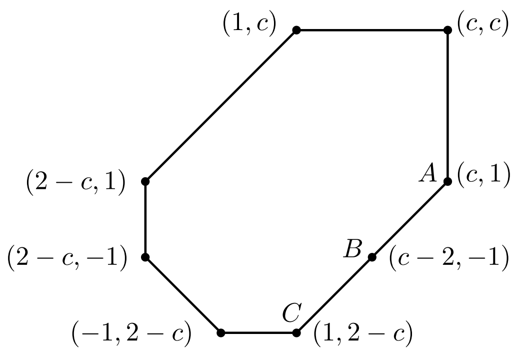

Figure 5. General orbit of (10) in the outer region of Γ $(c > 3)$.

after 5 mappings. Figure 6 (a) shows a typical mapping of the former and (b) the latter. Assume that k counts the number of cycles of mapping and P_k denotes a point on AC at the k-th cycle of a solution from an initial point P_0. Moreover, define r_k by

$$r_k = AP_k/AC.$$

By this definition, $0 < r_k < 1$ holds for any k. Moreover, r_k satisfies the following recurrence formula,

$$r_{k+1} = \begin{cases} r_k + 1 - \frac{2}{c-1} & (r_k < \frac{2}{c-1}) \\ r_k - \frac{2}{c-1} & (\text{otherwise}) \end{cases}.$$

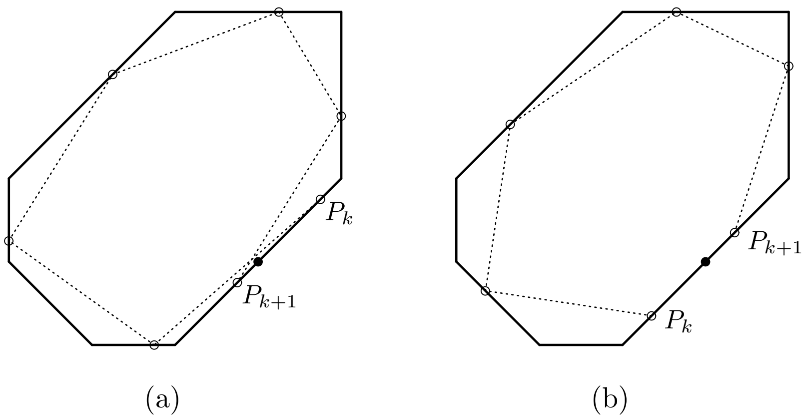

Figure 6. A sequence of mappings of a point on (a) AB, (b) BC.

This is a simple one dimensional dynamical system and we can easily see the solution is

$$r_k = \left\{ r_0 - \frac{2}{c-1} k \right\},$$

where $\{x\}$ denotes a fractional part of x. If $r_k = r_0$, that is, $P_k = P_0$, k must satisfy

$$\left\{ \frac{2}{c-1} k \right\} = 0 \quad \Leftrightarrow \quad \frac{2}{c-1} k \text{ is an integer.}$$

Therefore, if c is a rational number, k satisfying the above condition exists and the solution from P_0 becomes periodic. If not, $r_k \neq r_0$ ($P_k \neq P_0$) holds for any k.

Moreover, we can derive a period of solution from a value of c. If c is irrational, the period is ∞ according to the above discussion. If c is rational and is expressed by p/q where p and q are relatively prime integers, the period of solution is

$$\begin{cases} (5p-3q)/2 & (p \equiv q \bmod 2) \\ 5p - 3q & (\text{otherwise}) \end{cases}.$$

Similar results can be obtained for other cases, (9) with $A = -1$ and (3) with $\sigma = 1$ and 2. A period of solution is decided by each orbit and does not depend on the initial position of solution on the orbit. Solutions to (3) with $\sigma = 2$ are reported in the reference [6]. They are derived by ultradiscretizing the solutions to the original QRT system (2) including an elliptic function and the function taking fractional part also appears. Comparing with our results suggests there is a strong relation between geometric piecewise-linear dynamics and elliptic functions through ultradiscretization.

6. Concluding remarks

We studied integrable piecewise-linear mappings (3) obtained by ultradiscretizing the QRT system. In the case of $A = 0$, all solutions have the same period other than the fixed point. The mapping is expressed by a max function and is linearizable through the transformation of variables including a max function. Explicit solutions are also expressed by a max function using this linearizability.

In the case of $A \neq 0$, we showed a period of any solution on the same orbit is the same and it depends on the orbit. We can calculate the period from a parameter of the orbit by the function taking fractional part.

Finally we propose the following future problems. (i) Does a general class exist for linearizable piecewise-linear mappings? (ii) Can we obtain such a class by ultradiscretization of difference mappings? (iii) Is there an integrable piecewise-linear mapping with different periods depending on initial points on the same orbit?

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