Ultradiscrete Hamiltonian Systems

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Abstract

The method of ultradiscrete limit is applied to a series of discrete systems derived from Hamiltonian systems parametrized with corresponding lattice polygons. For every ultradiscrete system, general solution is obtained from the polar set of each lattice polygon.

1 Introduction

The non-analytic limit [1, 3]

$$\lim_{\epsilon\to+0}\epsilon\log\left(\exp\left[\frac{U}{\epsilon}\right]+\exp\left[\frac{V}{\epsilon}\right]+\cdots+\exp\left[\frac{W}{\epsilon}\right]\right)=\max(U,V,\ldots,W) \qquad (1)$$

has been introduced in the realm of soliton theory, which translates many of the full discrete soliton equations into the corresponding soliton cellular automata. [2, 4] Nowadays, this formula (1) is called the ultradiscrete limit, since the soliton equation of full discrete independent variables is further transformed through this limit into a system whose dependent variables are also discretized.

We propose applying the ultradiscrete limit to the Hamiltonian system

$$\frac{dQ}{dt}=\frac{\partial H}{\partial P},\quad \frac{dP}{dt}=-\frac{\partial H}{\partial Q}.$$

We will see that it's successful for some of the Hamiltonian functions in the following form,

$$H(Q,P)=\epsilon\log\left(\sum_{(j,k)\in\partial\Gamma\cap\mathbf{Z}^2}a_{(j,k)}\exp\left[\frac{jQ+kP}{\epsilon}\right]\right), \qquad (2)$$

where $\epsilon > 0$ is a parameter and the weight constants $a_{(j,k)}$ depend on a convex lattice polygon Γ having the origin as a unique internal lattice point. How to correspond each polygon to the respective Hamiltonian function is given concretely as follows.

1.1 Hamiltonian with Polygon Γ

Let Γ be a convex polygon such as

- every vertex $\in \mathbf{Z}^2$,
- having the origin $(0,0)$ as a unique internal lattice point.

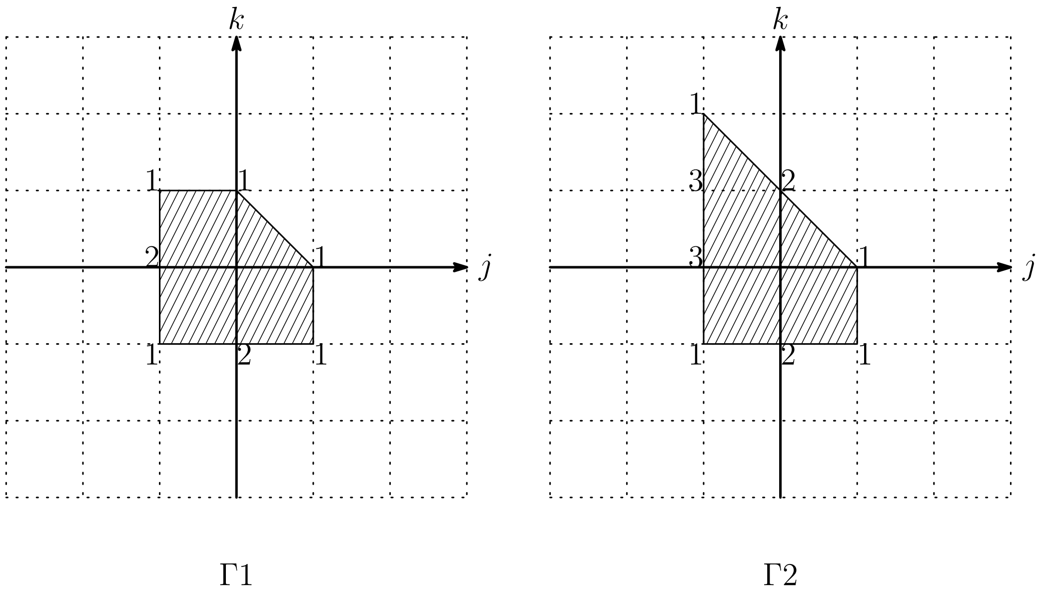

Figure 1: $\Gamma 1$ and $\Gamma 2$ with weight constants.

Figure 1 shows examples $\Gamma 1$ and $\Gamma 2$, where the labeled numbers on the lattice points of the edges are the weight constants. We correspond $\Gamma 1$ and $\Gamma 2$ to the following Hamiltonian functions:

$$H_{\Gamma 1}(Q,P) = \epsilon \log \left\{ \begin{array}{l} 1 \cdot e^{(1,0)\cdot(Q,P)/\epsilon} + 1 \cdot e^{(1,-1)\cdot(Q,P)/\epsilon} \\ +2 \cdot e^{(0,-1)\cdot(Q,P)/\epsilon} + 1 \cdot e^{(-1,-1)\cdot(Q,P)/\epsilon} \\ +2 \cdot e^{(-1,0)\cdot(Q,P)/\epsilon} + 1 \cdot e^{(-1,1)\cdot(Q,P)/\epsilon} \\ +1 \cdot e^{(0,1)\cdot(Q,P)/\epsilon} \end{array} \right\}, \tag{3}$$

$$H_{\Gamma 2}(Q,P) = \epsilon \log \left\{ \begin{array}{l} 1 \cdot e^{(1,0)\cdot(Q,P)/\epsilon} + 1 \cdot e^{(1,-1)\cdot(Q,P)/\epsilon} \\ +2 \cdot e^{(0,-1)\cdot(Q,P)/\epsilon} + 1 \cdot e^{(-1,-1)\cdot(Q,P)/\epsilon} \\ +3 \cdot e^{(-1,0)\cdot(Q,P)/\epsilon} + 3 \cdot e^{(-1,1)\cdot(Q,P)/\epsilon} \\ +1 \cdot e^{(-1,2)\cdot(Q,P)/\epsilon} + 2 \cdot e^{(0,1)\cdot(Q,P)/\epsilon} \end{array} \right\}. \tag{4}$$

We remark that the weight constants cannot be chosen arbitrarily, but have a considerable degree of freedom for our purpose of discretization. We use appropriate binomial coefficients as our weight constants for the present, which allow us to discretize the independent variable t. It would be a future study to determine the suitable general weight constants on general Γ. We give some more examples in the following section.

2 Discretization of Independent Variable t

We consider the discretization of our Hamiltonian systems.

First we note that our Hamiltonian functions $H = H_{\Gamma i}$ (for i=1,2) are strictly convex, accordingly, $\frac{\partial^2 H}{\partial Q^2} > 0$, $\frac{\partial^2 H}{\partial Q^2}\frac{\partial^2 H}{\partial P^2} - \left(\frac{\partial^2 H}{\partial Q \partial P}\right)^2 > 0$ for all $(Q, P) \in \mathbf{R}^2$, and they have a respective minimum value, accordingly, there exists respective unique finite point $(Q, P) = (Q_{fix}, P_{fix})$ such that $\frac{\partial H}{\partial Q} = \frac{\partial H}{\partial P} = 0$.

Now, we would make general consideration. Let $H(Q, P)$ be a Hamiltonian function given as strictly convex for all $(Q, P) \in \mathbf{R}^2$ and having a minimum value. Then, for arbitrary initial point (Q_0, P_0), the orbit $\{(Q(t), P(t)) | t \in \mathbf{R}, (Q(0), P(0)) = (Q_0, P_0)\}$ of the time evolution with

$$\frac{dQ}{dt} = \frac{\partial H}{\partial P}, \quad \frac{dP}{dt} = -\frac{\partial H}{\partial Q} \tag{5}$$

equals to the set $\{(Q, P) \in \mathbf{R}^2 | H(Q, P) = H(Q_0, P_0)\}$, that should be either some simple closed curve or the fixed point $\{(Q_{fix}, P_{fix})\}$. Here, we would emphasize that the time evolution $(Q(t), P(t))$ from $(Q(0), P(0)) = (Q_0, P_0)$ always passes over any (Q, P) such that $H(Q, P) = H(Q_0, P_0)$. This situation makes us sure how to obtain a discrete system.

The idea of discretization is simple: Let $(Q(t), P(t))$ be one of the representation of general solution to sys. (5), and $(X(t), Y(t))$ also be one,

$$\frac{dX}{dt} = \frac{\partial H}{\partial P}(X, Y), \quad \frac{dY}{dt} = -\frac{\partial H}{\partial Q}(X, Y), \tag{6}$$

as satisfying

$$H(Q, P) = H(X, Y). \tag{7}$$

Then, for any chosen initial point $(Q(0), P(0))$ of $(Q(t), P(t))$, there exists some point $(X(0), Y(0))$ as one of the initial point of $(X(t), Y(t))$ and exists some δ such that $(Q(\delta), P(\delta)) = (X(0), P(0))$ because of eq. (7) under that situation. Thus we have

$$E_\delta Q = X, \quad E_\delta P = Y, \tag{8}$$

where E_δ is time shift operator $E_\delta \phi(t) = \phi(t+\delta)$ with a properly chosen fixed parameter δ. We note that this parameter δ is determined respectively for each one-to-one correspondence,

$$(Q,P) \mapsto (X,Y) \quad = (F_\delta(Q,P), G_\delta(Q,P)). \tag{9}$$

Hence, we have found that, under that situation, the discrete system

$$E_\delta Q = F_\delta(Q,P), \quad E_\delta P = G_\delta(Q,P), \tag{10}$$

made of eqs. (8) and (9) allows the general solution $(Q(t), P(t))$ to sys. (5) with a properly chosen fixed parameter δ, if the couple of functions F_δ and G_δ satisfies the following condition,

$$\frac{dF_\delta}{dt} = \frac{\partial H}{\partial P}(F_\delta, G_\delta), \quad \frac{dG_\delta}{dt} = -\frac{\partial H}{\partial Q}(F_\delta, G_\delta), \quad H(Q,P) = H(F_\delta, G_\delta), \tag{11}$$

which is obtained with the use of eqs. (6), (7) and (9). (Remark: If some discrete system would have been given like as the form of sys. (10), the condition (11) would be the same as $\frac{d}{dt} \cdot E_\delta(Q,P) = E_\delta \cdot \frac{d}{dt}(Q,P), H(Q,P) = E_\delta H(Q,P)$.)

Here, we rewrite the condition (11) into more suitable form for our Hamiltonian functions. Let us introduce the following transformation,

$$\begin{array}{ll} & (Q(t), P(t), H(Q,P), F_\delta(Q,P), G_\delta(Q,P)) \\ \mapsto & (q(t), p(t), h(q,p), f(q,p), g(q,p)) := (e^{Q/\epsilon}, e^{P/\epsilon}, e^{H/\epsilon}, e^{F_\delta/\epsilon}, e^{G_\delta/\epsilon}), \end{array} \tag{12}$$

which transforms the condition (11) into the following system of equations,

$$\left\{ \begin{array}{c} \xi(q,p)\dfrac{\partial f}{\partial q} + \eta(q,p)\dfrac{\partial f}{\partial p} = \xi(f,g), \\ \xi(q,p)\dfrac{\partial g}{\partial q} + \eta(q,p)\dfrac{\partial g}{\partial p} = \eta(f,g), \\ h(q,p) = h(f,g), \end{array} \right. \tag{13}$$

where the functions ξ and η are defined as follows:

$$\xi(q,p) = qp\frac{\partial h(q,p)}{\partial p}, \quad \eta(q,p) = -qp\frac{\partial h(q,p)}{\partial q}.$$

The discrete system (10) is transformed into $E_\delta q = f(q,p)$, $E_\delta p = g(q,p)$.

2.1 Discretization for Case $\Gamma 1$

The transformation (12) transforms eq. (3) into

$$h_{\Gamma 1}(q,p) = q + \frac{q}{p} + \frac{2}{p} + \frac{1}{qp} + \frac{2}{q} + \frac{p}{q} + p,$$

and we have

$$\xi_{\Gamma 1}(q,p) = qp\frac{\partial h_{\Gamma 1}}{\partial p}, \quad \eta_{\Gamma 1}(q,p) = -qp\frac{\partial h_{\Gamma 1}}{\partial q}.$$

For this $\xi_{\Gamma 1}$ and $\eta_{\Gamma 1}$ above, we consider sys. (13),

$$\begin{cases} \xi_{\Gamma 1}(q,p)\frac{\partial f}{\partial q} + \eta_{\Gamma 1}(q,p)\frac{\partial f}{\partial p} = \xi_{\Gamma 1}(f,g), \\ \xi_{\Gamma 1}(q,p)\frac{\partial g}{\partial q} + \eta_{\Gamma 1}(q,p)\frac{\partial g}{\partial p} = \eta_{\Gamma 1}(f,g), \\ h_{\Gamma 1}(q,p) = h_{\Gamma 1}(f,g). \end{cases}$$

This system has an exact solution

$$f = p, \quad g = \frac{p+1}{q},$$

so, we have a rational mapping system

$$E_\delta q = p, \quad E_\delta p = \frac{p+1}{q},$$

which is transformed with eq. (12) into the discrete system

$$E_\delta Q = P, \quad E_\delta P = \epsilon \log(\exp[P/\epsilon] + 1) - Q,$$

which, for some δ, allows the general solution of continuous Hamiltonian system.

2.2 Discretization for Case Γ2

The transformation (12) transforms eq. (4) into

$$h_{\Gamma 2}(q,p) = q + \frac{q}{p} + \frac{2}{p} + \frac{1}{qp} + \frac{3}{q} + \frac{3p}{q} + \frac{p^2}{q} + 2p,$$

and we have

$$\xi_{\Gamma 2}(q,p) = qp\frac{\partial h_{\Gamma 2}}{\partial p}, \quad \eta_{\Gamma 2}(q,p) = -qp\frac{\partial h_{\Gamma 2}}{\partial q}.$$

For this $\xi_{\Gamma 2}$ and $\eta_{\Gamma 2}$ above, we consider sys. (13),

$$\begin{cases} \xi_{\Gamma 2}(q,p)\frac{\partial f}{\partial q} + \eta_{\Gamma 2}(q,p)\frac{\partial f}{\partial p} = \xi_{\Gamma 2}(f,g), \\ \xi_{\Gamma 2}(q,p)\frac{\partial g}{\partial q} + \eta_{\Gamma 2}(q,p)\frac{\partial g}{\partial p} = \eta_{\Gamma 2}(f,g), \\ h_{\Gamma 2}(q,p) = h_{\Gamma 2}(f,g). \end{cases}$$

This system has an exact solution

$$f = \frac{p(q+p+1)}{q}, \quad g = \frac{p+1}{q},$$

so we have a rational mapping system

$$E_\delta q = \frac{p(q+p+1)}{q}, \quad E_\delta p = \frac{p+1}{q},$$

which is transformed with eq. (12) into the discrete system

$$\begin{cases} E_\delta Q = P + \epsilon \log(\exp[Q/\epsilon] + \exp[P/\epsilon] + 1) - Q, \\ E_\delta P = \epsilon \log(\exp[P/\epsilon] + 1) - Q, \end{cases}$$

which, for some δ, allows the general solution of continuous Hamiltonian system.

2.3 Exact Solution to System (13), More Examples

We can make more examples. The following are some of them, where we denote only h_Γ and the respective exact solution f, g to every system of equations (13):

- $h_{\Gamma 3} = q + qp^{-1} + p^{-1} + q^{-1}p + p$.

$$f = \frac{p}{q(q+p+1)}, \quad g = \frac{1}{q+p}.$$

- $h_{\Gamma 4} = qp + 2p^{-1} + q^{-1}p^{-3} + 4q^{-1}p^{-2} + 6q^{-1}p^{-1} + 4q^{-1} + q^{-1}p + 2p$.

$$f = \frac{p(qp^2 + (p+1)^2)^2}{(p+1)^4}, \quad g = \frac{(p+1)^2}{qp^2}.$$

- $h_{\Gamma 5} = q + qp^{-1} + 2p^{-1} + q^{-1}p^{-1} + q^{-1} + p$.

$$f = \frac{qp}{q+1}, \quad g = \frac{qp+q+1}{q(q+1)}.$$

- $h_{\Gamma 6} = q + p^{-1} + q^{-1} + qp$.

$$f = \frac{p}{qp+1}, \quad g = \frac{1}{qp}.$$

- $h_{\Gamma 7} = qp + p^{-1} + q^{-1}p^{-1} + p + qp^2$.

$$f = qp^2(p+1), \quad g = \frac{1}{qp(p+1)}.$$

The following discussion is also applicable to them above.

3 Ultradiscrete Hamiltonian Systems

We apply the ultradiscrete limit $\epsilon \to +0$ to our Hamiltonian functions (3) and (4) by using eq. (1), then we obtain

$$\begin{aligned} H_{\Gamma 1}(Q,P) &= \max(Q, Q-P, -P, -Q-P, -Q, -Q+P, P), \\ H_{\Gamma 2}(Q,P) &= \max(Q, Q-P, -P, -Q-P, -Q, -Q+P, -Q+2P, P), \end{aligned}$$

and we obtain the following ultradiscrete systems,

$$\begin{array}{lll} E_\delta Q = P, \quad E_\delta P = \max(P,0) - Q, & \text{for} & H_{\Gamma 1}(Q,P), \\ E_\delta Q = P + \max(Q,P,0) - Q, \quad E_\delta P = \max(P,0) - Q, & \text{for} & H_{\Gamma 2}(Q,P). \end{array}$$

In order to find respective general solution to our ultradiscrete system above, we would use a concept of so called 'the polar set of Γ.' [5]

3.1 Polar Set of Γ

For a polygon Γ of the form defined in Section 1.1, the polar set of Γ is obtained in the following way: (This procedure is not available for general polygon.)

1. There exists a couple of edges next to the every vertex of the polygon Γ. For the every couple of edges, we construct a couple of outward normal vectors, where let the both vectors start from the vertex and end to each nearest lattice points.
2. Then, for every vertex, we construct the triangle spanned by the couple of outward normal vectors.
3. Finally, we construct a convex lattice polygon Γ^Δ by the parallel displacement of all triangles gathering to the origin. Then the polygon Γ^Δ is nothing but the polar set of Γ.

For our Γ1 and Γ2, we can construct the polar sets as fig. 2.

Hereafter, we treat the Hamiltonian function (2) as the ultradiscrete limit,

$$H_\Gamma(Q,P) = \max_{(j,k)\in\partial\Gamma\cap\mathbf{Z}^2} \{jQ + kP\}.$$

Then, we can show that the following formula

$$\{(Q,P) | H_\Gamma(Q,P) = 1\} = \partial\Gamma^\Delta, \tag{14}$$

holds for all Γ. This formula (14) means that the contour line $\{(Q,P)|H_\Gamma(Q,P) = 1\}$ of the ultradiscrete Hamiltonian function H_Γ corresponds to the boundary of the polar set Γ^Δ. Further, we note that the similarity $H_\Gamma(\lambda Q, \lambda P) = \lambda H_\Gamma(Q,P)$ $= \lambda$ always holds for all Γ and $\lambda \geq 0$. Therefore, we have obtained the procedure of constructing arbitrary contour line of the ultradiscrete Hamiltonian function, where we remark that any solution orbit of the ultradiscrete system should be on the contour line $\{(Q,P)|H_\Gamma(Q,P)=\lambda\}$ for some $\lambda \geq 0$ in the phase space.

We show the formula (14): Let Γ be N-gon with N-vertices (j_1,k_1), (j_2,k_2), ..., (j_N,k_N), where we set one of the vertices as $u_1 := (j_1,k_1)$, and set the other vertices as $u_j := (j_i,k_i)$ counterclockwise for $i=2,3,\ldots,N$, and we use cyclic notation $u_{N+1} := u_1$ or $u_0 := u_N$. We note,

$$\begin{aligned} H_\Gamma(Q,P) &= \max_{(j,k)\in\partial\Gamma\cap\mathbf{Z}^2}\{(jQ+kP)\} = \max_{(j,k)\in\partial\Gamma\cap\mathbf{Z}^2}\{(j,k)\cdot(Q,P)\} \\ &= \max_{1\leq i\leq N}\{u_i\cdot(Q,P)\}, \end{aligned}$$

because of the identity,

$$\max\{A\cdot(Q,P), B\cdot(Q,P), C\cdot(Q,P)\} = \max\{A\cdot(Q,P), C\cdot(Q,P)\},$$

which holds when B is on the line segment from A to C. Then, eq. (14) is reduced to $\{(Q,P)|\max_{1\leq i\leq N}\{u_i\cdot(Q,P)\}=1\} = \partial\Gamma^\Delta$. We show this equation. First we represent $\partial\Gamma$ with $u_i=(j_i,k_i)$. We recall Γ includes the origin $(0,0)$ as internal lattice point, and u_i have been set counterclockwise, so we have

$$\Gamma = \left\{(j,k)\,\middle|\det\begin{pmatrix} j_{i-1}-j & k_{i-1}-k \\ j_i-j & k_i-k \end{pmatrix} \geq 0, \text{ for } 1\leq i\leq N\right\}.$$

Here, we set

$$D_i := \det\begin{pmatrix} j_{i-1} & k_{i-1} \\ j_i & k_i \end{pmatrix},$$

for $1\leq i\leq N$. Then, we have $D_i>0$ and

$$\Gamma = \{(j,k)|(j,k)\cdot(k_i-k_{i-1}, j_{i-1}-j_i)\leq D_i, \text{ for } 1\leq i\leq N\}.$$

Thus we have

$$\partial\Gamma = \left\{(j,k)\,\middle|\max_{1\leq i\leq N}\{(j,k)\cdot(k_i-k_{i-1}, j_{i-1}-j_i)/D_i\}=1\right\}.$$

Now we set $v_i := (k_i-k_{i-1}, j_{i-1}-j_i)/D_i$, for $1\leq i\leq N$, that have the following properties:

- With $u_i\cdot v_i = u_{i-1}\cdot v_i = 1$, we have $(u_i-u_{i-1})\cdot v_i = 0$, hence, v_i is the normal for the edge (u_i-u_{i-1}) of Γ.
- We have
$$\det\begin{pmatrix} v_i \\ u_i-u_{i-1} \end{pmatrix} = \det\begin{pmatrix} k_i-k_{i-1} & j_{i-1}-j_i \\ j_i-j_{i-1} & k_i-k_{i-1} \end{pmatrix}/D_i > 0,$$
hence, v_i points outward.

- With $\gcd(j_{i-1}, k_{i-1}) = \gcd(j_i, k_i) = 1$, we have

  $$D_i = (\text{the number of lattice points on the edge from } u_{i-1} \text{ to } u_i) - 1,$$

  hence, $||v_i|| = ||u_i - u_{i-1}||/D_i$ is the length of the shortest lattice vector as the normal for $u_i - u_{i-1}$.

Therefore, we have found $\{v_i | 1 \leq i \leq N\}$ is the set of all vertices of Γ^Δ constructed by the procedure. Here, we note that

- Γ^Δ should have the unique internal lattice point $(0, 0)$.
- v_i should be counterclockwise for $i = 1, 2, \ldots, N$.
- Γ^Δ should be convex. (Because, $u_i \cdot (v_{i+1} - v_i) = 0$ for $i = 1, 2, \ldots, N$, and both $\{u_i\}$ and $\{v_i\}$ are counterclockwise.)

Secondly we represent $\partial\Gamma^\Delta$ with $v_i =: (Q_i, P_i)$. We can execute this as likewise as $\partial\Gamma$:

$$\partial\Gamma^\Delta = \left\{ (Q, P) \,\middle|\, \max_{1 \leq i \leq N} \{(P_{i+1} - P_i, Q_i - Q_{i+1}) \cdot (Q, P)/D'_i\} = 1 \right\},$$

where we have set

$$D'_i := \det \begin{pmatrix} Q_i & P_i \\ Q_{i+1} & P_{i+1} \end{pmatrix}.$$

Now, we set $w_i := (P_{i+1} - P_i, Q_i - Q_{i+1})/D'_i$, for $1 \leq i \leq N$, using cyclic notation $w_{n+1} := w_1$. Then, w_i becomes the outward normal for the edge $v_{i+1} - v_i$ of Γ^Δ, as the shortest lattice vector, and $\{w_i | 1 \leq i \leq N\}$ becomes the set of all vertices of a convex N-gon having the unique internal lattice point $(0, 0)$, where $(w_{i+1} - w_i) \cdot v_{i+1} = 0$, for $1 \leq i \leq N$, and we note such a convex N-gon should be uniquely determined. On the other hand, we already have found $(u_{i+1} - u_i) \cdot v_{i+1} = 0$. Hence, we obtain $w_i = u_i$ for $1 \leq i \leq N$, accordingly,

$$\partial\Gamma^\Delta = \{(Q, P) | \max_{1 \leq i \leq N} \{u_i \cdot (Q, P)\} = 1\} = \{(Q, P) | H_\Gamma(Q, P) = 1\}.$$

3.2 General Solution to Ultradiscrete System

We restrict the formula (14) on $\mathbf{Z}^2$:

$$\{(Q, P) \in \mathbf{Z}^2 | H_\Gamma(Q, P) = 1\} = \partial\Gamma^\Delta \cap \mathbf{Z}^2.$$

From this formula above, we can find the general solution to our ultradiscrete system as follows. (We call 'general solution' in the meaning that the solution includes any time evolution solution from arbitrary chosen initial point $(Q_0, P_0) \in \mathbf{R}^2$.)

First, we note that our ultradiscrete system for Γ1,

$$E_\delta Q = P, \quad E_\delta P = \max(P, 0) - Q,$$

causes a permutation of the following set:

$$\begin{aligned}\{(Q, P) \in \mathbf{Z}^2 | H_{\Gamma 1}(Q, P) = 1\} &= \partial\Gamma 1^{\Delta} \cap \mathbf{Z}^2 \\ &= \{(1, 0), (0, -1), (-1, 0), (0, 1), (1, 1)\}.\end{aligned}$$

Actually, we have

$$(1, 0) \mapsto (0, -1) \mapsto (-1, 0) \mapsto (0, 1) \mapsto (1, 1) \mapsto (1, 0).$$

Here, we make $\Phi(Q_0, P_0)$ denote the periodic solution to our system for Γ1 of an initial value (Q_0, P_0) out of $\partial\Gamma 1^{\Delta} \cap \mathbf{Z}^2$. For instance,

$$\Phi(1, 0) = \begin{cases} (1, 0), & t/\delta \equiv 0 \pmod 5, \\ (0, -1), & t/\delta \equiv 1 \pmod 5, \\ (-1, 0), & t/\delta \equiv 2 \pmod 5, \\ (0, 1), & t/\delta \equiv 3 \pmod 5, \\ (1, 1), & t/\delta \equiv 4 \pmod 5. \end{cases}$$

Then, the general solution to our system for Γ1 is written in the following form:

$$\mu\Phi(Q_0, P_0) + \nu E_\delta \Phi(Q_0, P_0), \tag{15}$$

for arbitrary $\mu \geq 0, \ \nu \geq 0$ and $(Q_0, P_0) \in \partial\Gamma 1^{\Delta} \cap \mathbf{Z}^2$. This solution is, indeed, general solution, since,

$$\begin{aligned}\mathbf{R}^2 = \quad & \{\mu(1, 0) + \nu(0, -1)\} \cup \{\mu(0, -1) + \nu(-1, 0)\} \cup \{\mu(-1, 0) + \nu(0, 1)\} \\ & \cup \{\mu(0, 1) + \nu(1, 1)\} \cup \{\mu(1, 1) + \nu(1, 0)\},\end{aligned}$$

which is to say, the arbitrary point of the phase space $\mathbf{R}^2$ can be taken as the initial point of, where the one-step time evolution of our ultradiscrete system for Γ1 causes a permutation of the five components of $\mathbf{R}^2$ decomposed as the above.

Second, we note that our ultradiscrete system for Γ2,

$$E_\delta Q = P + \max(Q, P, 0) - Q, \quad E_\delta P = \max(P, 0) - Q.$$

causes a permutation of the following set:

$$\begin{aligned}\{(Q, P) \in \mathbf{Z}^2 | H_{\Gamma 2}(Q, P) = 1\} &= \partial\Gamma 2^{\Delta} \cap \mathbf{Z}^2 \\ &= \{(1, 0), (0, -1), (-1, 0), (1, 1)\}.\end{aligned}$$

Actually, we have

$$(1, 0) \mapsto (0, -1) \mapsto (-1, 0) \mapsto (1, 1) \mapsto (1, 0).$$

Here, we make $\Psi(Q_0, P_0)$ denote the periodic solution to our system for $\Gamma 2$ of an initial value (Q_0, P_0) out of $\partial\Gamma 2^{\Delta} \cap \mathbf{Z}^2$. For instance,

$$\Psi(1,0) = \begin{cases} (1,0), & t/\delta \equiv 0 \pmod 4, \\ (0,-1), & t/\delta \equiv 1 \pmod 4, \\ (-1,0), & t/\delta \equiv 2 \pmod 4, \\ (1,1), & t/\delta \equiv 3 \pmod 4. \end{cases}$$

Then, the general solution to our system for $\Gamma 2$ is written in the following form:

$$\mu\Psi(Q_0, P_0) + \nu E_\delta \Psi(Q_0, P_0), \tag{16}$$

for arbitrary $\mu \geq 0$, $\nu \geq 0$ and $(Q_0, P_0) \in \partial\Gamma 2^{\Delta} \cap \mathbf{Z}^2$.

Both general solutions (15) and (16) are represented in the same form. In the other cases of $\Gamma 3$, $\Gamma 4$, …, and so on, the general solutions are represented in the similar form without some arrangement of basic periodic solutions for superposition.

4 Summary

The method of ultradiscrete limit is applied to the discrete systems derived from the Hamiltonian systems as parametrized with lattice polygons. For every ultradiscrete systems, the general solution is obtained from the polar set of each lattice polygon.

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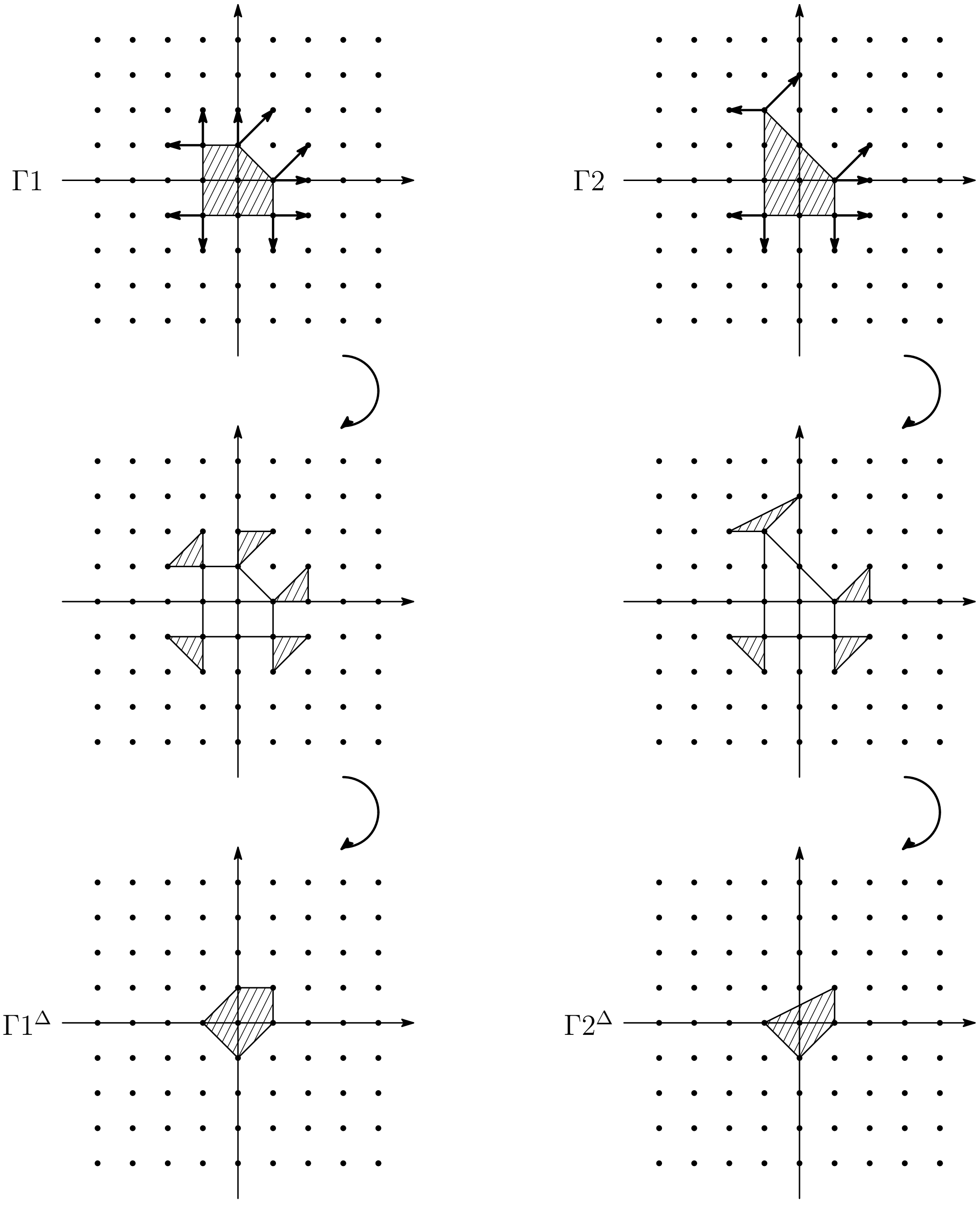

Figure 2: How to make the polar set Γ^{Δ}.